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MATH FACTOR

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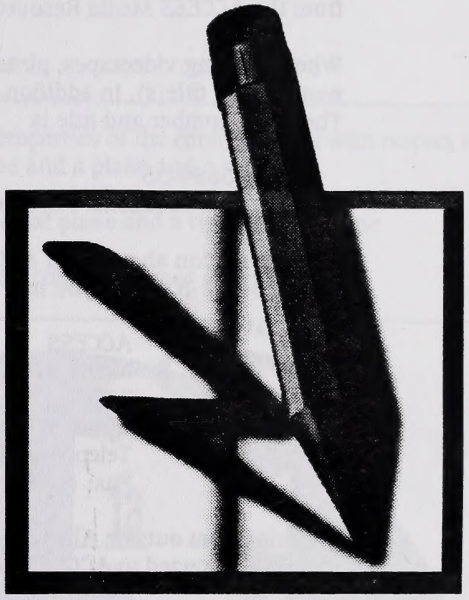
QUADRATIC RELATIONS



Lesson 1:
Physical Properties of Conic Sections

Student Workbook

Produced by

THE EDUCATION STATION

for

Alberta EDUCATION

This guide accompanies **Physical Properties of Conic Sections** from the Math Factor series.

Writer: Roxann Trouth
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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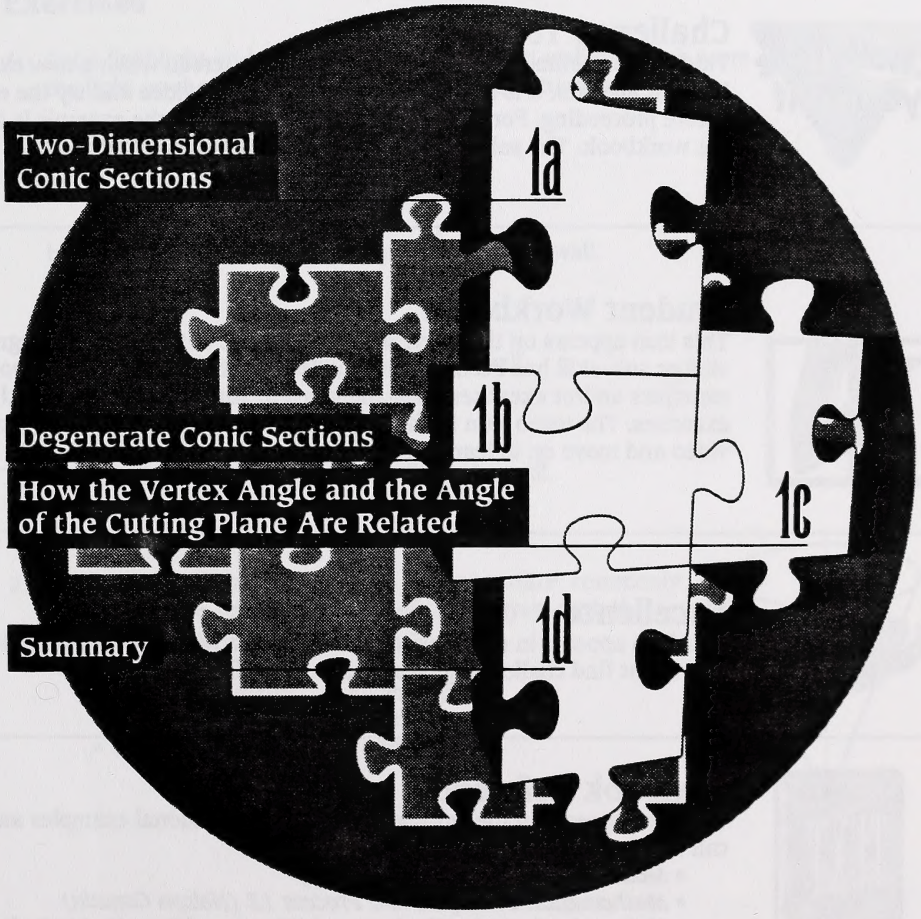
Lesson 1: PHYSICAL PROPERTIES OF CONIC SECTIONS

LEARNER EXPECTATIONS

By the end of Lesson 1 you should be able to

- show you understand the physical properties of the conic sections with respect to the intersection of a plane and a cone and a plane and a cylinder
- describe the conic formed by a particular plane and a right circular cone
- identify the point at which each of the conics becomes degenerate

This lesson has four parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

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- challenge yourself to find the solution before it is given
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Two-Dimensional Conic Sections

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Watch the Introduction and Video Segment 1a.

You can generate your own conics by cutting cross sections of a styrofoam or plasticine cone. Observe the results when you make each of the following:

- vertical cut or a cut perpendicular to the base of the cone
- horizontal cut or a cut parallel to the base of the cone
- slant cut or a cut parallel to the side of the cone

Exercises

1. Get a flashlight and shine a beam of light on a wall in a dark room. What happens to the light on the wall in each of the following situations?

   a. The beam is perpendicular to the wall.

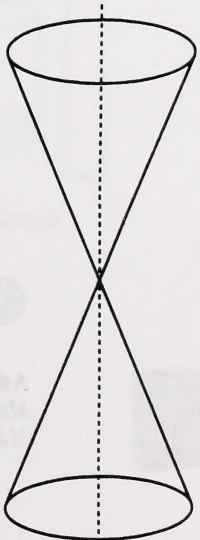
   b. The beam is held at an oblique angle to the wall.

   c. The flashlight is moved closer to the wall.

   d. The flashlight is moved away from the wall.

2. The diagram illustrates a circular conical surface, commonly called a double-napped cone. Label the diagram with the following terms:

   - axis of symmetry
   - generator
   - upper nappe
   - lower nappe
   - vertex
   - vertex angle



3. Describe how to intersect a double-napped cone with a plane to obtain the following:

   a. a circle

   b. an ellipse

   c. a parabola

   d. a hyperbola

4. Explain how a circle is a limiting case of the ellipse.

5. a. Who was the first person to use double-napped cones for conic sections?

   b. Where and when was this done?

EXERCISE 1a SOLUTIONS are at the back of this workbook.

TEXT

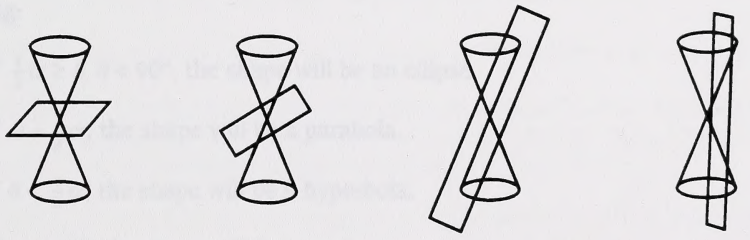
Additional Practice
Mathematics 12: pp. 90-93, p. 94 (Question 4)
Mathematics: Principles and Process 12: p. 337 (Question 1)

Degenerate Conic Sections

Watch Video Segment 1b.

There are five degenerate conic sections: a point, a straight line, two intersecting straight lines, two parallel lines and no graph (also called no locus).

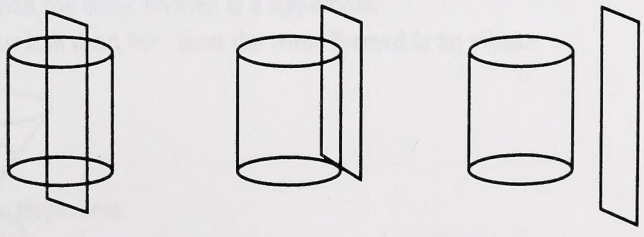
Using the diagrams below, one can see that both a circle and an ellipse degenerate to a point. A parabola degenerates to a single straight line, two parallel lines or no graph, while a hyperbola degenerates to two intersecting lines. The following diagrams show how a plane intersects a double-napped cone to produce a degenerate case of each of the conic sections.



Circle **Ellipse** **Parabola** **Hyperbola**

When the vertex is extended infinitely far away from the circular base of the cone, a **cylinder** results.

If the angle of the cutting plane is parallel to the generator, a **parabola** will degenerate in one of three ways.



Two Parallel Lines **One Line** **No Graph**

Exercises

6. Describe when the degenerate case for each of the following conic sections occurs.

   a. a circle or an ellipse

   b. a hyperbola

   c. a parabola

EXERCISE 1b SOLUTIONS are at the back of this workbook.

How the Vertex Angle and the Angle of the Cutting Plane Are Related

MATH FACTOR

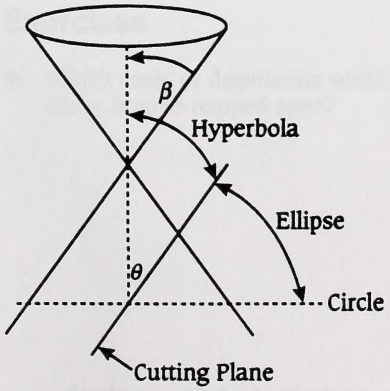
Watch Video Segment 1c.

If the vertex angle is of measure α, then the angle between the generator and the axis of symmetry will be $\frac{1}{2}\alpha$. If the acute angle between the cutting plane and the axis of symmetry is labelled θ, then the relationship between $\frac{1}{2}\alpha$ and θ determines the shape of the conic formed as long as the cutting plane does not pass through the vertex. Look at the following:

- If $\frac{1}{2}\alpha \geq \theta$, $\theta < 90°$, the shape will be an ellipse.
- If $\theta = \frac{1}{2}\alpha$, the shape will be a parabola.
- If $\theta < \frac{1}{2}\alpha$, the shape will be a hyperbola.
- If $\theta = 90°$, the shape will be a circle.

In the diagram below, β, equivalent to $\frac{1}{2}\alpha$ mentioned previously, is the angle that a generator makes with the axis of symmetry and θ is the angle that the cutting plane makes with the axis of symmetry.

- If $\theta = \beta$, then the conic formed is a parabola.
- If $\theta < \beta$, then the conic formed is a hyperbola.
- If $\theta > \beta$ but less than 90°, then the conic formed is an ellipse.



Exercises

7. If the plane does not go through the vertex of a double-napped cone, then name the conic formed by the intersection of a plane and a double-napped cone in each of the following cases.

	Measure of the Vertex Angle (α)	Angle of the Cutting Plane with the Axis of Symmetry (β)	Name of the Conic Section
a.	48°	22°	
b.	32°	35°	
c.	18°	90°	
d.	60°	30°	

8. The measure of the angle formed by the cutting plane and the axis of symmetry of a double-napped cone is 40°. Calculate the range of measures of the vertex angle that will result in the following:

   a. a parabola

   b. a hyperbola

   c. an ellipse

   d. a circle

   e. a point

EXERCISE 1c SOLUTIONS are at the back of this workbook.

Summary

There are two ways to describe the intersection of a plane and a double-napped cone:

- The angle between the plane and the axis of symmetry is compared with the angle between a generator and the axis of symmetry.
- The angle between the plane and the circular base is compared with the angle between a generator and the circular base.

If both the angle of the cutting plane and the angle of the generator are measured from the base, then the following may be stated:

- If the cutting plane is parallel to the circular base, a **circle** is formed.
- If the angle between the cutting plane and the circular base is **less** than the angle between a generator and the circular base, an **ellipse** is formed. The word *ellipse* comes from the Greek word *ellipsis* meaning *a falling short.*
- If the angle between the cutting plane and the circular base is **equal** to the angle between a generator and the circular base, that is, if the plane is parallel to the generator a **parabola** is formed. The word *parabola* comes from the Greek word meaning *comparison* or *juxtaposition.* (Compare with the words *parallel, compare* or *parable.*)
- If the angle between the cutting plane and the circular base is **greater** than the angle between a generator and the circular base, a **hyperbola** is formed. The plane cuts both nappes; thus, the hyperbola always consists of two parts. The word *hyperbola* comes from the Greek word meaning *excess.* (Compare with the words *hyperbole* or *hyperactive.*)

Exercises

9. Which conic or degenerate conic cannot be generated by the intersection of a plane and a double-napped cone?

10. Which conic must have a double-napped cone in order to be generated by the intersection of a plane and a conical surface?

11. a. Describe how the intersection of a plane and a double-napped cone could result in two intersecting lines?

    b. What is this called?

12. Describe the movement of a plane cutting a double-napped cone to show how an ellipse degenerates to a point.

13. What condition(s) must be satisfied so that the intersection of a plane and a cone results in a parabola?

14. For each double-napped cone shown, list the positive angle θ (or range of angles) between the cutting plane and the axis of symmetry, to obtain the conic sections listed in the chart on the next page. State any other conditions, besides the size of θ, that would also be necessary for these conics to occur.

Figure 1

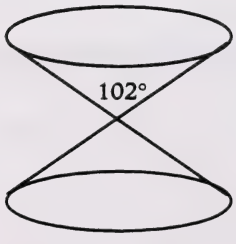


Figure 2

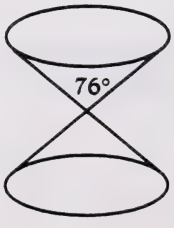
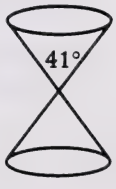


Figure 3



Shape	Figure 1 ($\alpha = 102°$)	Figure 2 ($\alpha = 76°$)	Figure 3 ($\alpha = 41°$)
Circle			
Ellipse			
Parabola			
Hyperbola			
Point			
Line			
Two Intersecting Lines			

EXERCISE 1d SOLUTIONS are at the back of this workbook.

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. In subsequent lessons, you will learn other ways to describe the different conic sections, and you will learn how the understanding of conic sections can be applied to areas such as astronomy and architecture.

Solutions

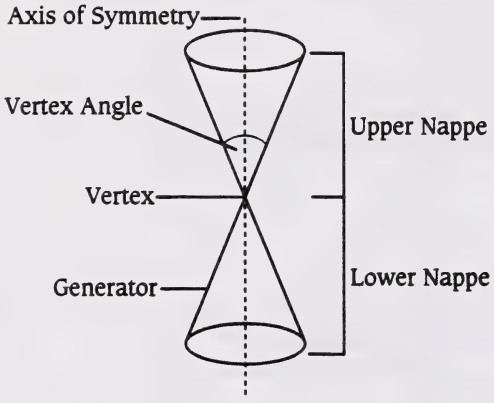
EXERCISE 1a

1. a. The flashlight acts like the vertex of a cone, and the wall intersects the light beam at an angle of 90°; therefore, a circle of light results.

   b. Because the wall intersects the light beam at an angle, an ellipse results.

   c. The cone of light is intersected closer to the vertex, so the shape is smaller.

   d. The cone of light is intersected farther from the vertex, so the shape is larger.

2. 

3. a. Intersect the cone with a plane at 90° to the axis of symmetry, not through the vertex.

   b. Intersect the cone with a plane at an angle less than 90° but greater than the angle the generator makes with the axis of symmetry, not through the vertex.

   c. Intersect the cone with a plane at an angle equal to the angle that the generator makes with the axis of symmetry, not through the vertex.

   d. Intersect the cone with a plane at an angle less than the angle that the generator makes with the axis of symmetry, not through the vertex.

4. As the angle of the cutting plane gets closer to being parallel to the base of the cone, the ellipse becomes more and more round, until it eventually becomes a circle.

5. a. The mathematician Apollonius was the first person to use double-napped cones for conic sections.

   b. This was done in Greece in 200 B.C.

EXERCISE 1b

6. a. A **circle** and an **ellipse** will both degenerate to a point when the cutting plane passes through the vertex of the double-napped cone but at an angle greater than the angle that the generator makes with the axis of symmetry.

   b. A **hyperbola** will degenerate to **two intersecting lines** when the cutting plane passes through the vertex but at an angle less than the angle that the generator makes with the axis of symmetry.

c. A **parabola** will degenerate to a **single line** when the cutting plane passes through the vertex at an angle equal to the angle that the generator makes with the axis of symmetry and, therefore, cuts along the generator itself.

When the vertex of the cone is extended to infinity, a cylinder results. If the cutting plane passes through the cylinder, the degenerate of **two parallel lines** results; if the cutting plane falls along the edge of the cylinder (along the generator), the degenerate of a **single straight line** results, and if the cutting plane falls outside the cylinder, the degenerate of **no graph** will result.

EXERCISE 1c

7. a. $\alpha = 48^\circ$, $\alpha + 2 = 24^\circ$, $22^\circ < 24^\circ$; therefore, a hyperbola is obtained.

   b. $\alpha = 32^\circ$, $\alpha + 2 = 16^\circ$, $35^\circ > 16^\circ$; therefore, an ellipse is obtained.

   c. 90° to the axis of symmetry; therefore, a circle is obtained.

   d. $\alpha = 60^\circ$, $\alpha + 2 = 30^\circ$, $30^\circ = 30^\circ$; therefore, a parabola is obtained.

8. a. $\theta = 40°$; therefore, the vertex angle $\alpha = 2 \times 40 = 80°$.

   b. The vertex angle must be greater than 80° ($\alpha > 80°$).

   c. $\alpha < 80°$

   d. A circle cannot occur. The angle of the cutting plane must be 90°.

   e. A point occurs when the plane passes through the vertex; the size of the vertex angle may vary, but must be less than 80°. If $\alpha = 80°$, the cutting plane will lie along the generator, and a straight line will result. If $\alpha > 80°$, two intersecting lines will result.

EXERCISE 1d

9. Two parallel lines can only be generated if the **vertex** of the double-napped cone is **stretched to infinity** so that a **cylinder** results.

10. A **hyperbola**, or its degenerate of **two intersecting lines**, requires both nappes in order to occur.

11. a. Two intersecting lines occur if the cutting plane passes **through the vertex** at an angle **greater** than the acute angle that the generator makes with the circular base of the double-napped cone.

    b. This is called a degenerate hyperbola.

12. An ellipse is obtained by the intersection of a double-napped cone with a plane at an angle less than the angle between the generator and the circular base, **not through the vertex**. If the plane moves closer and closer to the vertex, the ellipse becomes smaller and smaller. When the plane intersects the vertex, the ellipse becomes a single point.

13. The plane must intersect the cone **parallel** to the generator and **not through the vertex**.

14.

Shape	Figure 1 ($\alpha = 102°$)	Figure 2 ($\alpha = 76°$)	Figure 3 ($\alpha = 41°$)
Circle	$\theta = 90°$, plane not through the vertex	$\theta = 90°$, plane not through the vertex	$\theta = 90°$, plane not through the vertex
Ellipse	$51° < \theta < 90°$, plane not through the vertex	$38° < \theta < 90°$, plane not through the vertex	$20.5° < \theta < 90°$, plane not through the vertex
Parabola	$\theta = 51°$, plane not through the vertex	$\theta = 38°$, plane not through the vertex	$\theta = 20.5°$, plane not through the vertex
Hyperbola	$0° \leq \theta < 51°$, plane not through the vertex	$0° \leq \theta < 38°$, plane not through the vertex	$0° \leq \theta < 20.5°$, plane not through the vertex
Point	$51° < \theta \leq 90°$, plane not through the vertex	$38° < \theta \leq 90°$, plane not through the vertex	$20.5° < \theta \leq 90°$, plane not through the vertex
Line	$\theta = 51°$, plane through the vertex	$\theta = 38°$, plane through the vertex	$\theta = 20.5°$, plane through the vertex
Two Intersecting Lines	$0° \leq \theta < 51°$, plane through the vertex	$0° \leq \theta < 38°$, plane through the vertex	$0° \leq \theta < 20.5°$, plane through the vertex

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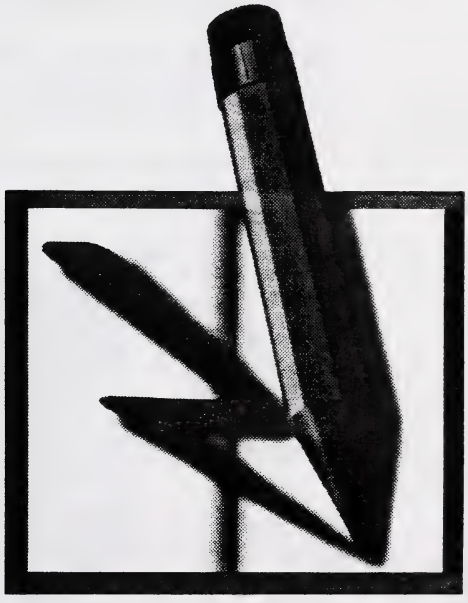
QUADRATIC RELATIONS



Lesson 2:
General Quadratic Relation:
An Introduction

Student Workbook

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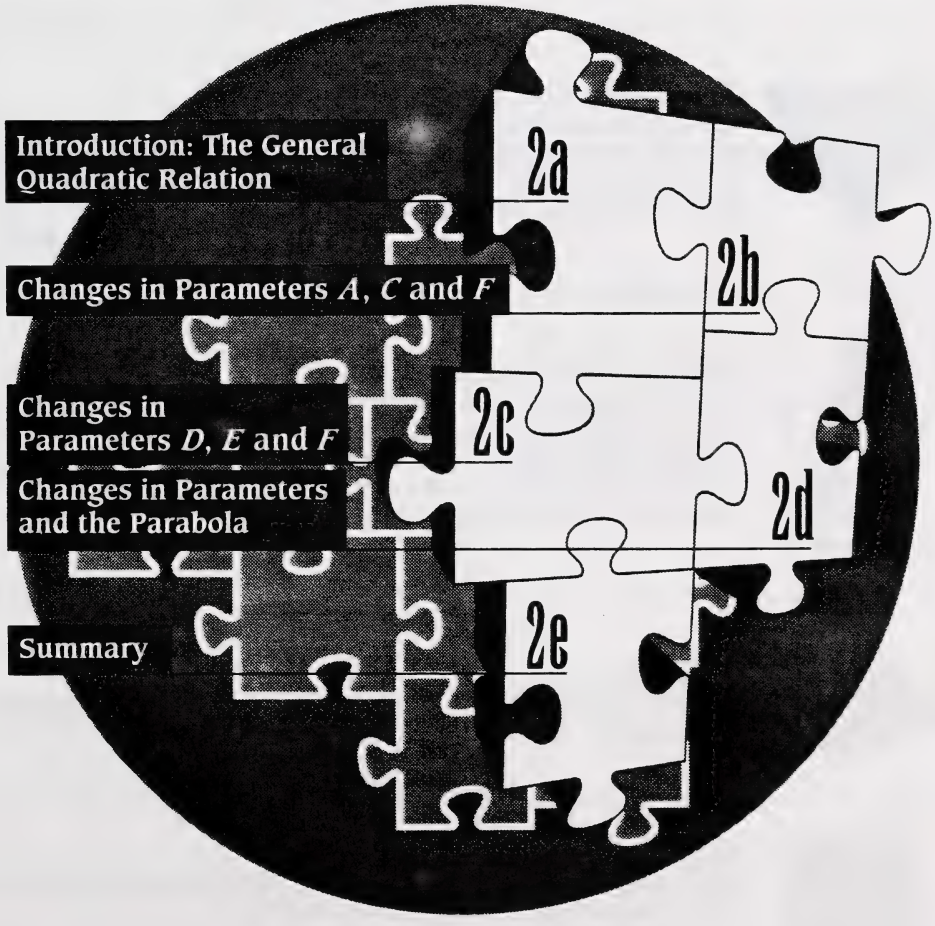
Lesson 2: GENERAL QUADRATIC RELATION: AN INTRODUCTION

LEARNER EXPECTATIONS

By the end of Lesson 2 you should be able to

- demonstrate an understanding of the general quadratic relation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$, as the algebraic representation of a conic

This lesson has five parts:



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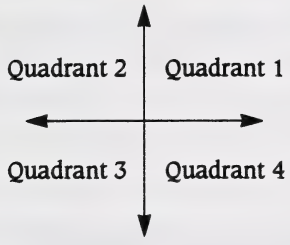
Building Blocks

Do you remember the relative values of integers?

Recall that the integer 5 is greater than 3, but the integer -5 is less than -3. Therefore, -25 is considered to be a smaller number than -15; so, a **larger negative** is actually considered to be a **smaller number**.

Do you remember how the quadrants of a coordinate plane are numbered?

Take a few moments to study the following diagram.



Do you remember how to determine the x- and y-intercepts of the graph of a relation?

Example

Determine the x- and y- intercepts of the graph of the quadratic relation $x^2 + 4y^2 - 16 = 0$.

Solution

To determine the x-intercepts, substitute $y = 0$ into the equation.

$$\begin{aligned} x^2 + 4(0)^2 - 16 &= 0 \\ x^2 - 16 &= 0 \\ x^2 &= 16 \\ x &= \pm\sqrt{16} \\ x &= \pm 4 \end{aligned}$$

To find the y-intercepts, substitute $x = 0$ into the equation.

$$\begin{aligned} (0)^2 + 4y^2 - 16 &= 0 \\ 4y^2 &= 16 \\ y^2 &= 4 \\ y &= \pm\sqrt{4} \\ y &= \pm 2 \end{aligned}$$

Introduction: The General Quadratic Relation

Watch the Introduction and Video Segment 2a.

The equation of a quadratic relation is one that has at least one second-degree term. The general form of the equation of a quadratic relation is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. The numerical coefficients A, B, C, D, E and F are also called parameters. There are some restrictions on these parameters.

- A, B, C, D, E and F must be members of the set of real numbers.
- In this course, the parameter B is equal to 0.
- A and C cannot both be equal to 0. If A and C both equal 0, the relation no longer has a second-degree term; therefore, the relation becomes linear.

Example

Rewrite the equation $(x+5)^2 = 5 - y^2$ in general form.

Solution

Expand and simplify; then add and/or subtract to get all terms on one side of the equal sign.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$$(x+5)^2 = 5 - y^2$$ Expand.

$$x^2 + 10x + 25 = 5 - y^2$$

$$x^2 + 10x + 25 - 5 + y^2 = 0$$ Get all the terms on one side.

$$x^2 + y^2 + 10x + 20 = 0$$ Add the like terms.

Remember, the degree of an algebraic expression is the degree of the highest-degree term. For example, to find the degree of the term x^2y, add the exponents of all the variables, $2 + 1 = 3$. The expression x^2y has a degree of three.

Exercises

1. Which of the following equations are the equations of quadratic relations?

   a. $x^2 - 2y = 5$

   b. $x^3 + 2x^2 + 3x = 0$

   c. $xy = 10$

   d. $3x - 2y = 11$

   e. $x^2 y - 3y^2 + x - 5 = 0$

   f. $x = y^2$

2. Rewrite the following equations in the general form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

   a. $y = x^2 - 2$

   b. $x^2 + y^2 = 38$

   c. $3x^2 + 2x - 5 = y^2 - 2y + 1$

   d. $4 - x - 5y^2 = 2x + x^2 - 10$

e. $(2x+1)^2 = (3-2y)^2$

3. Fill in the blanks for each of the answers obtained in question 2.

a. $A =$ ____, $B =$ ____, $C =$ ____, $D =$ ____, $E =$ ____, $F =$ ____

b. $A =$ ____, $B =$ ____, $C =$ ____, $D =$ ____, $E =$ ____, $F =$ ____

c. $A =$ ____, $B =$ ____, $C =$ ____, $D =$ ____, $E =$ ____, $F =$ ____

d. $A =$ ____, $B =$ ____, $C =$ ____, $D =$ ____, $E =$ ____, $F =$ ____

e. $A =$ ____, $B =$ ____, $C =$ ____, $D =$ ____, $E =$ ____, $F =$ ____

EXERCISE 2a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 94 (Question 1), p. 98 (Question 4), p. 107 (Question 2)

Changes in Parameters *A*, *C* and *F*

Watch Video Segment 2b.

To obtain the nondegenerate cases of the conic sections, the values of the numerical coefficients in the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ must satisfy certain conditions. These conditions are as follows:

- A circle is formed when $A = C$.
- An ellipse is formed when $AC > 0$ and $A \neq C$.
- A hyperbola is formed when $AC < 0$.
- A parabola is formed when $A = 0$ or $C = 0$.

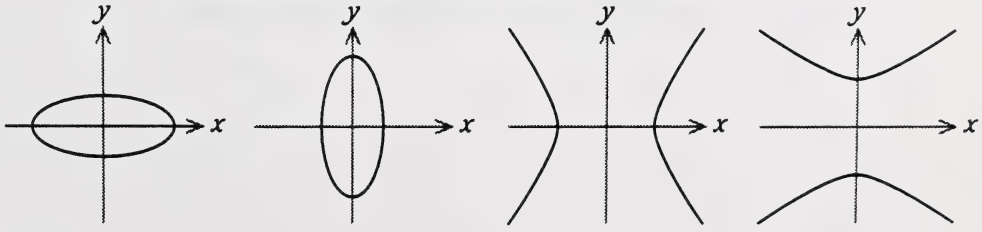
If all other coefficients remain unchanged, changing F affects the size and width of the conic section. Changing F may also cause the degenerate case of a conic section to occur.

If the graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, $B = 0$ is:

- a circle or an ellipse centred at the origin, then $A = C$ or $AC > 0$ and $F < 0$.
- a point, then $A = C$ or $AC > 0$ and $F = 0$.
- a hyperbola centred at the origin, then $AC < 0$ and $F \neq 0$.
- two intersecting lines, then $AC < 0$ and $F = 0$.

Sometimes a hyperbola or an ellipse is said to be horizontal or vertical. The following diagrams demonstrate what is meant by this terminology:

Horizontal Ellipse **Vertical Ellipse** **Horizontal Hyperbola** **Vertical Hyperbola**



You can determine whether the hyperbola or the ellipse is horizontal or vertical by graphing their equations or by looking at their equations and comparing pairs of parameters.

For the ellipse, look at the parameters A and C. If $A > C$, then the ellipse is vertical; if $A < C$, then the ellipse is horizontal.

For the hyperbola centred at the origin, look at the parameters A and F. If A and F are both positive or both negative, then the graph will have no x-intercepts and the hyperbola is vertical. If A is positive and F is negative (or vice versa), then the graph will have x-intercepts and the hyperbola is horizontal.

Exercises

4. The equation of a quadratic relation is $x^2 + Cy^2 - 16 = 0$. Describe the type of quadratic relation formed for the following values of C:

   a. $C = 4$

   b. $C = 1$

   c. $C = 0.25$

   d. $C = 0$

   e. $C = -4$

5. The equation of a quadratic relation is $x^2 + y^2 + F = 0$. Describe the type of quadratic relation formed for the following values of F:

   a. $F = -16$

   b. $F = -4$

   c. $F = 0$

   d. $F = 4$

6. The equation of a quadratic relation is $x^2 - y^2 + F = 0$. Describe the type of quadratic relation formed for the following values of F:

   a. $F = -16$

   b. $F = -4$

   c. $F = 0$

   d. $F = 4$

7. Describe the graph of each of the following equations.

   a. $5x^2 - 5y^2 - 10 = 0$

   b. $x^2 + 4y^2 - 25 = 0$

   c. $y^2 - 14 = x^2$

   d. $2x^2 + 2y^2 = 0$

   e. $x^2 - y - 20 = 0$

   f. $x^2 - 5 + y^2 + 4 = 0$

8. Compare and contrast the graphs that are described by the following equations.

   a. $6x^2 + y^2 - 20 = 0$ and $x^2 + 6y^2 - 5 = 0$

   b. $x^2 + y^2 - 10 = 0$ and $x^2 - y^2 - 10 = 0$

   c. $x^2 + y^2 - 20 = 0$ and $5x^2 + 5y^2 - 20 = 0$

EXERCISE 2b SOLUTIONS are at the back of this workbook.

Changes in Parameters *D*, *E* and *F*

Watch Video Segment 2c.

For the circle, ellipse and hyperbola, changes in the parameters D and E affect the location of the centres of the conics, hence, their placement on the coordinate plane.

The following generalizations may be made for the circle, ellipse and hyperbola if only the parameters D and E change and $A > 0$.

- If $D > 0$, the centre shifts left.
- If $D < 0$, the centre shifts right.
- If $E > 0$, the centre shifts down for the circle and ellipse but shifts up for the hyperbola.
- If $E < 0$, the centre shifts up for the circle and ellipse but shifts down for the hyperbola.

Exercises

9. Identify the quadratic relation described by each of the following equations. Then determine which quadrant its centre would lie if the quadratic relation were graphed.

a. $x^2 + 5y^2 + 10x - 15y - 20 = 0$

b. $3x^2 - 5y^2 - 8x + 12y - 20 = 0$

c. $2x^2 - 5x + 2y^2 + 6y - 40 = 0$

d. $2x^2 - 3x + 7y + 5y^2 - 49 = 0$

10. Describe the graph of the quadratic relation defined by $x^2 + 3y^2 - 4x - 8y - 2 = 0$.

11. Determine the equation of a circle that has its centre in quadrant 2.

12. If $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is a hyperbola with its centre at the origin, then describe the restrictions on the coefficients.

13. Given the equation $x^2 + 4y^2 - 6x + 5y - 25 = 0$, answer the following:

    a. What type of curve does this equation describe?

    b. When graphed, what quadrant is the centre located?

    c. If the value of E was -5, in which quadrant would the centre be located?

    d. How would the graph change if the value of F was -49?

14. The equation $9x^2 + Cy^2 - 4x - 2y - 36 = 0$ represents an ellipse. What value of C would make this ellipse a circle?

15. Given the equation $2x^2 - 5y^2 - 3x + 7y - 49 = 0$, describe how its graph changes if the following occur:

    a. the values of A and C are interchanged

    b. the values of D and E are interchanged

EXERCISE 2c SOLUTIONS are at the back of this workbook.

Changes in Parameters and the Parabola

Watch Video Segment 2d.

The equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ describes a parabola if either A or C equals 0.

To obtain a nondegenerate case of a parabola, it is necessary to have a second-degree x-term and a first-degree y-term, **or** a second-degree y-term and a first-degree x-term. In terms of the coefficients, a parabola exists if

- $C = 0$ and $A \neq 0$ and $E \neq 0$
- $A = 0$ and $C \neq 0$ and $D \neq 0$

If $C = 0$, $A \neq 0$ and $E \neq 0$, the parabola will be vertical, or it will open up or open down. Changes in the values of A and E affect the width and direction of the opening of the parabola, while changes in the values of D and F affect the location of the vertex.

If $A = 0$, $C \neq 0$ and $D \neq 0$, the parabola will be horizontal, or it will open to the left or to the right. Changes in the values of C and D affect the width and direction of the opening of the parabola, while changes in the values of E and F affect the location of the vertex.

The direction of the opening of a parabola can be determined by graphing or by examining its equation. By comparing parameters in the equation, you will find the following:

- If A and E have different signs, the parabola will be vertical and will open up.
- If A and E have the same sign, the parabola will be vertical and will open down.
- If C and D have different signs, the parabola will be horizontal and will open to the right.
- If C and D have the same sign, the parabola will be horizontal and will open to the left.

Exercises

16. Given the equation of a parabola $3x^2 + 8x - 6y - 15 = 0$, answer the following:

    a. In which direction does the parabola open?

    b. If the value of A were increased, what would happen to the shape of the graph?

c. If A were negative, which way would the graph open?

17. Given the equation of a parabola $y^2 - 10x + 5y - 6 = 0$, answer the following:

a. In which direction does the parabola open?

b. If the values of E and F were 0, what would the effect be on the graph?

18. Given the equation of a parabola $x^2 + y - 9x + 20 = 0$, answer the following:

a. In which direction does the parabola open?

b. Which coefficient(s) must be changed and what values must they be assigned in order for the parabola to open another way?

EXERCISE 2d SOLUTIONS are at the back of this workbook.

Summary

Watch Video Segment 2e.

Which quadratic relation described by the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$, depends on the values of A and C. When $B = 0$, the axes of symmetry are parallel to the x- and y-axes. Changes in the value of D result in a horizontal shift of the centre of the graph while changes in the value of E result in a vertical shift of the centre. Changes in the value of F may cause the quadratic relation to degenerate to a point, two intersecting lines or no graph.

Exercises

19. For each of the following equations, determine the quadratic relation and, where possible, the orientation and the location of the centre of its graph.

    a. $x^2 + y^2 + 4x + 6y - 20 = 0$

    b. $4x^2 + y^2 + 4x + 8y - 3 = 0$

    c. $x^2 + 4x + 6y - 1 = 0$

    d. $7x^2 - 4y^2 + 14x + 2y - 15 = 0$

    e. $9x^2 + 16y^2 - 49x - 144 = 0$

    f. $-5x^2 - 5y^2 + 48x + 36y + 4 = 0$

    g. $x^2 - 5x + 4 = 0$

    h. $y^2 + 4x + 6y - 20 = 0$

20. If $x^2 + 4y^2 + Dx + Ey - 16 = 0$ describes an ellipse with its centre in quadrant 3, then determine values of D and E.

21. Given the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$, determine the conditions that must be satisfied if this equation describes the following:

    a. a circle

    b. a vertical ellipse

    c. a horizontal hyperbola centred at the origin

    d. a parabola that opens down and with vertex at $(0, -5)$

EXERCISE 2e SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: pp. 143-144 (Questions 2, 4, 8), pp. 147-148 (Questions 1, 2, 3)
Holtmath 12: p. 180 (Questions 1, 3)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectation at the beginning of this lesson and check it off if you have mastered it. In subsequent lessons, you will learn how to analyze the graph of a conic section and obtain a quadratic equation that will represent the graph.

Solutions

EXERCISE 2a

1. a. Yes
   b. No (A degree 3 term is present.)
   c. Yes (The term xy is degree 2.)
   d. No (This is a linear relation.)
   e. No (The term x^2y is degree 3.)
   f. Yes

2. a. $x^2 - y - 2 = 0$
   b. $x^2 + y^2 - 38 = 0$
   c. $3x^2 - y^2 + 2x + 2y - 6 = 0$
   d. $0 = x^2 + 5y^2 + 3x - 14$
   e. $4x^2 - 4y^2 + 4x + 12y - 8 = 0$

3. a. $A = 1, B = 0, C = 0, D = 0, E = -1, F = -2$
   b. $A = 1, B = 0, C = 1, D = 0, E = 0, F = -38$
   c. $A = 3, B = 0, C = -1, D = 2, E = 2, F = -6$
   d. $A = 1, B = 0, C = 5, D = 3, E = 0, F = -14$
   e. $A = 4, B = 0, C = -4, D = 4, E = 12, F = -8$

EXERCISE 2b

4. a. an ellipse
   $A \neq C$; they are both the same sign $(AC > 0)$.
   b. a circle
   $A = C$
   c. an ellipse (See question 4a.)
   d. two parallel lines
   The equation becomes $x^2 - 16 = 0$, which factors to $(x - 4)(x + 4) = 0$. The solutions are $x = 4$ or $x = -4$, two vertical lines.
   e. a hyperbola
   $A \neq C$, and A and C are opposite signs $(AC < 0)$.

5. a. a circle
   $A = C$ and $F < 0$.
   b. a smaller circle
   (Same as question 5a, but F is a smaller negative.)

c. a point
If $F = 0$, the equation becomes $x^2 + y^2 = 0$, which has only one solution, $x = 0$ and $y = 0$.

d. no graph
No real values satisfy $x^2 + y^2 + 4 = 0$. If any real number is squared, the result is a nonnegative number. The sum of two nonnegative numbers and 4 can never equal 0.

6. a. a horizontal hyperbola
$AC < 0$, $x^2 - y^2 - 16 = 0$ has x-intercepts of 4 and -4 and no y-intercepts.

b. a horizontal hyperbola, but the branches are closer together (x-intercepts are 2 and -2.)

c. two intersecting lines
$x^2 - y^2 = 0$ factors to $(x - y)(x + y) = 0$ and has two solutions, $x = y$ and $x = -y$.
Both are linear relations.

d. a vertical hyperbola
The equation becomes $x^2 - y^2 + 4 = 0$. Now there are no x-intercepts, and there are y-intercepts of 2 and -2.

7. a. a hyperbola
$AC < 0$

b. an ellipse
$A \neq C$, $AC > 0$

c. a vertical hyperbola
Rewrite as $x^2 - y^2 + 14 = 0$.

d. a point
Degenerate of a circle, $F = 0$.

e. a parabola, opens up
Rewrite as $y = x^2 + 20$.

f. a small circle
Rewrite as $x^2 + y^2 - 1 = 0$.

8. a. Both are ellipses with centres on the origin. Find the x- and y-intercepts to show that the first ellipse is vertical and the second is horizontal. The first ellipse is larger than the second. (F is more negative.)

b. The first is a circle and the second is a hyperbola. Both are centred on the origin and have the same x-intercepts.

c. Both circles centred on the origin; the second is smaller than the first. Divide every term of the second equation by 5 to get $x^2 + y^2 - 4 = 0$.

EXERCISE 2c

9. a. an ellipse, centred in quadrant 2

b. a hyperbola, centred in quadrant 1

c. a circle, centred in quadrant 4
Rearrange the terms in order of degree to get $2x^2 + 2y^2 - 5x + 6y - 40 = 0$.

d. an ellipse, centred in quadrant 4
Rearrange the terms in order of degree to get $2x^2 + 5y^2 - 3x + 7y - 49 = 0$.

10. This is the graph of a horizontal ellipse with its centre in the first quadrant.

11. $A = C$, $D > 0$ and $E < 0$
There are many answers. One example is $x^2 + y^2 + 5x - 3y - 20 = 0$.

12. $AC < 0$, $D = 0$, $E = 0$ and $F \neq 0$

13. a. an ellipse

b. quadrant 4

c. quadrant 1

d. It would increase in size.

14. $C = 9$

15. a. The hyperbola will change from horizontal to vertical.

b. The centre of the hyperbola will change from quadrant 1 to quadrant 3.

EXERCISE 2d

16. a. up

b. The graph would become narrower.

c. down

17. a. to the right

b. The parabola would have its vertex on the origin.

18. a. down

b. Change A to a negative value or change E to a negative value and the graph will open up.

EXERCISE 2e

19. a. circle, centre quadrant 3

b. vertical ellipse, centre quadrant 3

c. parabola, opens down

d. horizontal hyperbola, centre quadrant 2

e. horizontal ellipse, centre on the positive x-axis

f. circle, centre quadrant 1 (Change all the signs so A is positive.)

g. degenerate of a parabola, two parallel vertical lines

h. parabola, opens left

20. The graph must move left and down; therefore, both D and E should have positive values. F is not positive, so a degenerate will not occur.

21. a. A must equal C; if F is negative, D and E can be negative or positive.
For example, $x^2 + y^2 + 5x - 8y - 15 = 0$.

b. A must be greater than C and both must have the same sign. $D = 0$.
For example, $4x^2 + y^2 - 10 = 0$.

c. If A is positive, then C and F must be negative. If A is negative, then both C and F must be positive. $D = 0$ and $E = 0$. For example, $x^2 - 3y^2 - 17 = 0$.

d. $A \neq 0$, $E \neq 0$ and $A \times E > 0$. $C = 0$, $D = 0$. The y-intercept is -5.
For example, $x^2 + y + 5 = 0$.

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MATH FACTOR

QUADRATIC RELATIONS

Lesson 3:
General Quadratic Relation: Circles and Ellipses

Student Workbook

Produced by

THE EDUCATION STATION

ACCESS

for

Alberta EDUCATION

This guide accompanies **General Quadratic Relation: Circles and Ellipses** from the Math Factor series.

Writer: Roxann Trouth
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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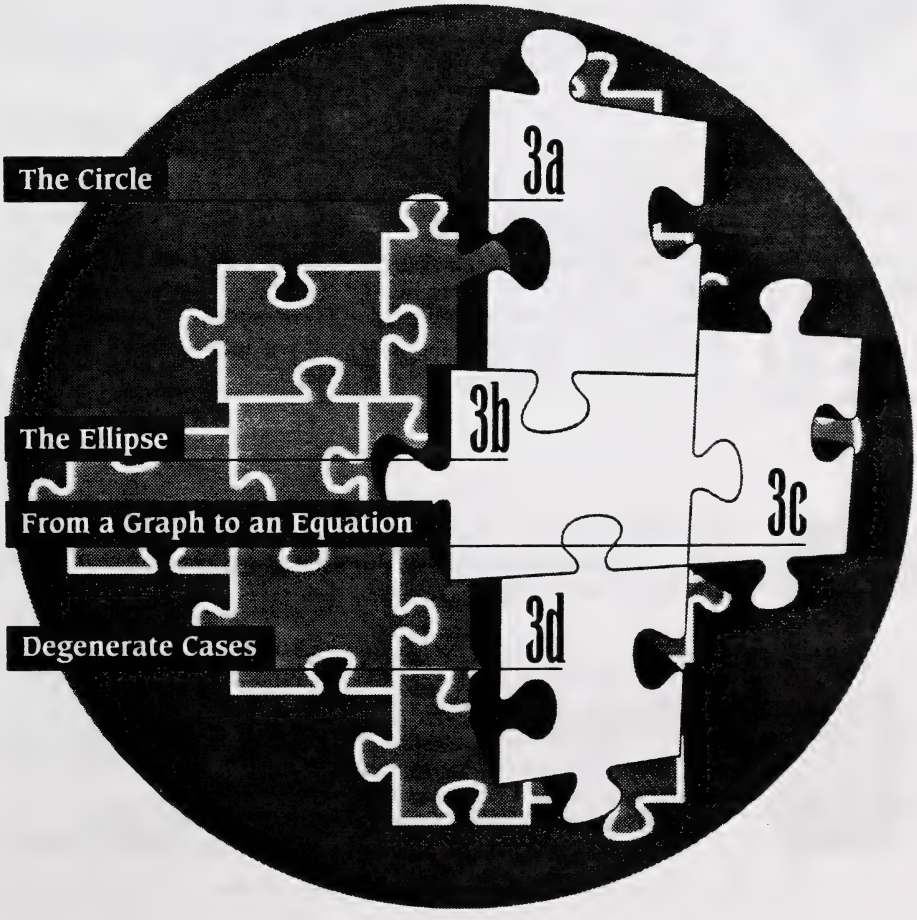
Lesson 3: GENERAL QUADRATIC RELATION: CIRCLES AND ELLIPSES

LEARNER EXPECTATIONS

By the end of Lesson 3 you should be able to

- describe the conics generated by various combinations of values for the numerical coefficients, in particular, circles and ellipses
- investigate and describe the effects of the numerical coefficients on the graphs of circles and ellipses using calculators or computers
- recognize that designating $B = 0$ gives an axis of symmetry parallel to one of the coordinate axes
- analyze the graphs of ellipses, given their equations
- recognize the conditions required for an ellipse to become a circle
- investigate and describe the effects of the numerical coefficients on the orientation, size and shape of the graphs of circles and ellipses

This lesson has four parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

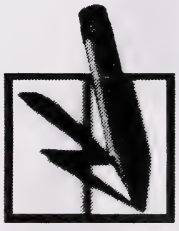
Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.



Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

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This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Building Blocks

Always Make *A* Positive

When studying the role of the parameters A, B, C, D, E and F in the quadratic relation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, keep the value of A positive. If A has a negative value, simply multiply both sides of the defining equation by -1. This does not change the graph of the relation.

Example 1

Analyze the equation $-x^2 - y^2 + 2x - 10y + 20 = 0$.

Solution

First multiply every term by -1 so that the coefficient A is positive.

$$(-1)\left(-x^2 - y^2 + 2x - 10y + 20\right) = 0$$
$$x^2 + y^2 - 2x + 10y - 20 = 0$$

This equation describes a circle with its centre in quadrant 4.

The Radius of a Circle

If the equation of a circle is written as $x^2 + y^2 + F = 0$, where $F < 0$, then the radius of the circle will be the square root of the absolute value of F. For example, in the equation $x^2 + y^2 - 100 = 0$, the radius of the circle will be $\sqrt{100}$ or 10 units.

Example 2

Find the radius of the circle given by the equation $2x^2 + 2y^2 - 36 = 0$.

Solution

Divide every term by 2 to get $x^2 + y^2 - 18 = 0$.

The radius is $\sqrt{18}$ or $3\sqrt{2}$ or approximately 4.2 units.

Use of the Technology

When examining the characteristics of the circle and ellipse, it is helpful if you have access to a graphing calculator or computer graphing program. If neither of these is available to you, refer to Lesson 2 where you will find alternative methods for determining the effects of changing the parameters A, C, D, E and F.

The computer program used in the video is Zap-a-Graph® and may be obtained from the Alberta Distance Learning Centre.

The Circle

Watch the Introduction and Video Segment 3a.

A circle is a collection of all points in a plane at a given distance (called the radius) from a fixed point in the plane (called the centre).

Using the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$, a circle centred at the origin exists when $A = C$, $B = 0$, $D = E = 0$ and $F < 0$.

When the values of A, C and F remain the same and a value of D exists, then the circle will be affected in the following ways:

- The centre will move from the origin to a position either to the left or to the right of the origin.
- The radius will be smaller or larger.

When the values of A, C and F remain the same and a value of E exists, then the circle will be affected in the following ways:

- The centre will move from the origin to a position either above or below the origin.
- The radius will be smaller or larger.

When the values of A, C, D and E remain the same and the value of F changes, then the radius of the circle will be smaller or larger. For some values of F, a degenerate case of the circle may be obtained.

Exercises

1. Determine what happens to the graph of $x^2 + y^2 - 40 = 0$ when the following occurs:

   a. the value of F is decreased to -50

   b. the value of F is increased to -3

   c. the value of D is changed from 0 to $+20$, and the value of F remains at -40

d. the value of E is changed from 0 to -5, and the value of F remains at -40

2. Answer the following given the quadratic equation $x^2 + y^2 + F = 0$.

   a. What happens to the values of D and F if the centre of the graph is shifted to the left of the origin and there is no change in the radius of the circle?

   b. What happens to the values of E and F if the centre of the graph is shifted above the origin and there is no change in the radius of the circle?

   c. What value of F will result in a degenerate case of a circle?

EXERCISE 3a SOLUTIONS are at the back of this workbook.

The Ellipse

Using the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$, an ellipse centred at the origin exists when $A \neq C$, $AC > 0$, $D = E = 0$ and $F < 0$.

When the values of A, C and F remain the same and a value of D exists, then the ellipse will be affected in the following ways:

- The centre will move from the origin to a position either to the left or to the right of the origin.
- The width or length will change.

When the values of A, C and F remain the same and a value of E exists, then the ellipse will be affected in the following ways:

- The centre will move from the origin to a position either above or below the origin.
- The width or length will change.

When the values of A, C, D and E remain the same and the value of F changes, then the length and width of the ellipse will change. For some values of F, a degenerate case of the ellipse may be obtained.

Exercises

3. Why can circles and ellipses both be described as closed curves?

4. The equation $x^2 + y^2 - 2x + 5y - 30 = 0$ describes a circle. Determine the coefficient(s) that must change and how they must change in order to transform the circle into the following:

   a. a vertical ellipse

b. a horizontal ellipse, centred on the origin

5. Given the ellipse $4x^2 + y^2 - 16 = 0$, describe the changes needed to the coefficients in order to get the following:

a. an ellipse stretched horizontally

b. a larger ellipse

c. a smaller ellipse, with its centre shifted to the right of the origin

d. a rounder horizontal ellipse

e. a larger ellipse centred in quadrant 3

f. an ellipse with x-intercepts of 1 and -1 and y-intercepts of 8 and -8

EXERCISE 3b SOLUTIONS are at the back of this workbook.

From a Graph to an Equation

Watch Video Segment 3c.

When given the graph of a quadratic relation, you can predict its equation by following these steps using the general equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

Step 1: Set $B = 0$ to keep the orientation of the axes of symmetry of the quadratic relation parallel to the coordinate grid.

Step 2: Determine a relationship between A and C.

- If the graph is a circle, then set $A = C$.
- If the graph is a vertical ellipse, then set $A > C$ and $AC > 0$.
- If the graph is a horizontal ellipse, then set $A < C$ and $AC > 0$.

Step 3: If a rounder ellipse is needed, make the values of A and C closer in value. If a more elongated ellipse is needed, increase the difference between the values of A and C.

Step 4: Determine values for D and E.

- If the centre of the quadratic relation is at the origin, then set $D = 0$ and $E = 0$.
- If the centre of the quadratic relation is to the left of the origin, then set $D > 0$.
- If the centre of the quadratic relation is to the right of the origin, then set $D < 0$.
- If the centre of the quadratic relation is below the origin, then set $E > 0$.
- If the centre of the quadratic relation is above the origin, then set $E < 0$.

Step 5: Select a value of F, which is negative to start with, so that the equation can be graphed initially. Then use a graphing calculator or a computer graphing program to graph your equation.

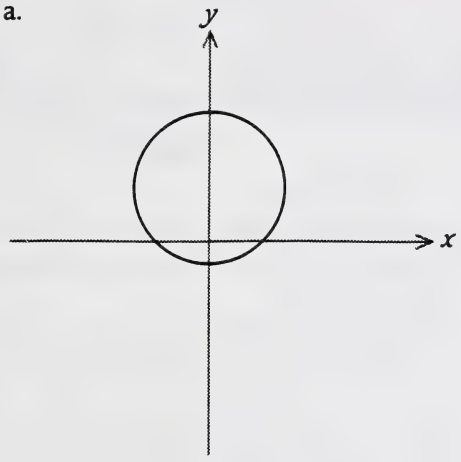
Step 6: Increase or decrease the values of D and E, if necessary, in order to position the centre (appropriately). Remember that changing D and E will also affect the size of the graph.

Step 7: If the position is approximately correct, but the size is not, make F a larger negative to increase the size of the graph or a smaller negative (or even a positive) to decrease the size of the graph.

Exercises

6. Predict an equation for each of the following quadratic relations.

a.



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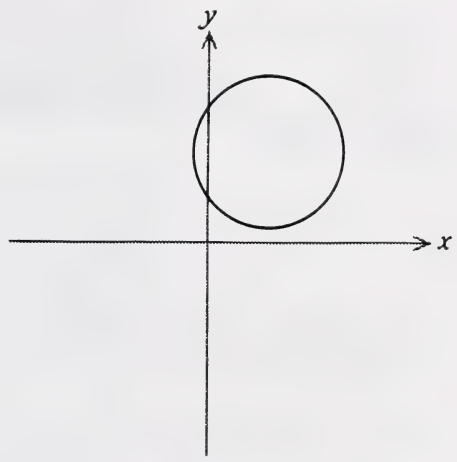

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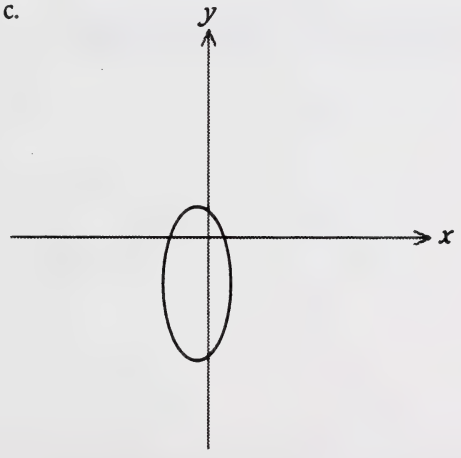

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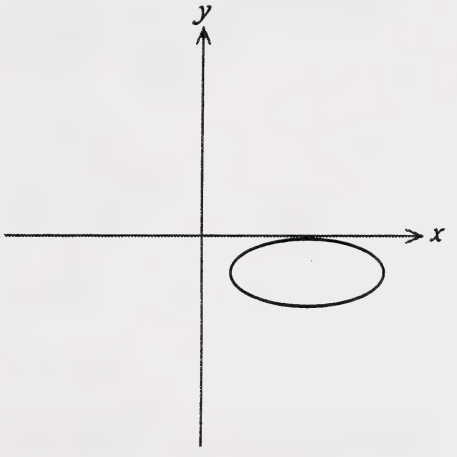
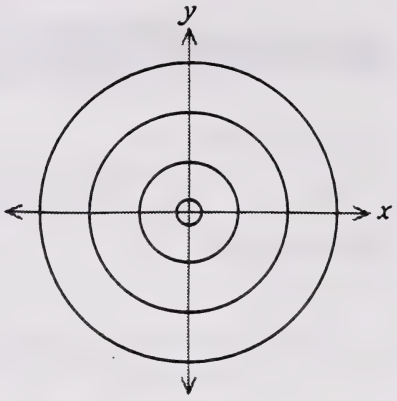


Note: There are many different values of the coefficients that will produce graphs similar to the preceding graphs. The solutions provided in this lesson are only one set of possible answers. If technology is not available, suggest values for A, B, C, D and E that might result in a graph like each of the ones shown, and explain why each value was chosen.

7. A curling house has a centre line and a tee line that cross at the centre of the house. The graph at the right represents the rings of a curling house. The largest circle has a diameter of about twelve feet; the next largest circle has a diameter of eight feet; the next one has a diameter of four feet; and the smallest circle, called the button, has a diameter of one foot.



Explain how you would create such a pattern using equations of a circle. Give the initial equation and explain what you would change in the equation to create all the circles. Also, explain why you chose the particular values for A, B, C, D, E and F.

EXERCISE 3c SOLUTIONS are at the back of this workbook.

Degenerate Cases

For the equation of a circle or an ellipse, given by $Ax^2 + Cy^2 + F = 0$, the graph degenerates to a point if $F = 0$ and degenerates to no graph if $F > 0$.

Note: In Lesson 1, p. 5, the diagram illustrates the circle degenerating into a single point. However, it is not possible to draw a diagram showing the degenerate form of no graph. This is because it is not possible to draw a circle with a radius that is not a real number.

If either D or E, or both D and E, are not equal to 0, then there exists some positive value of F such that the conic degenerates to a point; and if F is greater than that value, the conic degenerates to no graph.

Exercises

8. Given the equation $Ax^2 + Cy^2 + F = 0$, $AC > 0$, which parameter could cause the degenerate of a point to occur? What value must it have?

9. Change the value of F in the equation $x^2 + y^2 - 3x - 5y + 15 = 0$ so that a circle is obtained.

10. The limiting case of an ellipse is a circle. Does this mean that a circle is a degenerate case of an ellipse? Explain why or why not.

11. Is it possible to have a positive value of F in the equation of a quadratic relation where the graph is a circle or an ellipse? Explain why or why not.

12. Which is the only degenerate that cannot be demonstrated by cutting a double-napped cone or a cylinder?

EXERCISE 3d SOLUTIONS are at the back of this workbook.

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. In the next lesson, you will learn how to determine the degenerates of hyperbolas and ellipses.

Solutions

EXERCISE 3a

1. a. The circle becomes larger. The radius increases from $\sqrt{40}$ to $\sqrt{50}$.

   b. The circle becomes smaller. The radius decreases from $\sqrt{40}$ to $\sqrt{3}$.

   c. The centre of the graph shifts to the left and the circle increases in size. The y-intercepts remain the same.

   d. The centre of the graph shifts upwards and the circle increases in size. The x-intercepts remain the same.

2. a. If the graph is shifted left but the size remains the same, then the value of D will be positive and the value of F will increase (to a smaller negative or possibly to a positive value).

   b. The value of E will become negative and the value of F will increase.

   c. For the relation $x^2 + y^2 + F = 0$, a value of $F = 0$ will result in a point, and any positive value of F will result in no graph.

EXERCISE 3b

3. Circles and ellipses can be described as closed curves because they enclose a distinct interior.

4. a. Either A must be increased so that it is greater than C (e.g., $A = 5$), or C must be decreased so that it is less than A but still positive (e.g., $C = 0.25$).

   b. The values of D and E must both be changed to 0, and either C must be increased to a value greater than A, or A must be decreased to a value less than C but still positive. For example, $x^2 + 4y^2 - 30 = 0$ or $0.25x^2 + y^2 - 30 = 0$.

5. a. Increase the difference in value between A and C as in $10x^2 + y^2 - 16 = 0$ or $4x^2 + 0.25y^2 - 16 = 0$.

   b. Decrease the value of F (make it a larger negative). An example is $4x^2 + y^2 - 40 = 0$.

   c. Add an x-term with a negative coefficient and increase the value of F. An example is $4x^2 + y^2 - 5x - 1 = 0$.

   d. Decrease the value of A and increase the value of C until C is just a little larger than A such as $x^2 + 2y^2 - 16 = 0$.

   e. Add an x-term with a positive coefficient and a y-term with a positive coefficient, and either keep the value of F the same or decrease it depending on how much bigger the ellipse is to be. $4x^2 + y^2 + 3x + 5y - 16 = 0$ is an example.

f. To find the new value of A, which will result in x-intercepts of 1 and -11, set $y = 0$. The new equation becomes the following:

$$\begin{aligned} Ax^2 - 16 &= 0 \\ A(1)^2 - 16 &= 0 \quad \text{(if one } x\text{-intercept is 1)} \\ A - 16 &= 0 \quad \text{(Simplify.)} \\ A &= 16 \end{aligned}$$

To find the new value of C, which will result in y-intercepts of 8 and -8, set $x = 0$. The new equation becomes the following:

$$\begin{aligned} Cy^2 - 16 &= 0 \\ C(8)^2 - 16 &= 0 \quad \text{(if one } y\text{-intercept is 8)} \\ 64C - 16 &= 0 \quad \text{(Simplify.)} \\ 64C &= 16 \\ C &= \frac{16}{64} \\ &= 0.25 \end{aligned}$$

In summary, the coefficient A must be changed to 16, and the coefficient C must be changed to 0.25.

EXERCISE 3c

6. Please note that all quadratic relations shown have axes of symmetry parallel to the x- and y-axes; therefore, in all cases, $B = 0$.

a. This is a circle shifted up from the origin; therefore, there is an E value but no D value. Start with $A = 1$, $B = 0$, $C = 1$, $D = 0$, $E = -5$ and $F = -20$. Try graphing the equation $x^2 + y^2 - 5y - 20 = 0$. This one looks about right.

b. This is a circle with the centre shifted up and to the right; therefore, there are both D and E values, both negative. Since the graph is shifted slightly more vertically than it is horizontally, the E value should be a slightly larger negative than the D value. Start with $A = 1$, $B = 0$, $C = 1$, $D = -4$, $E = -6$ and $F = -10$. The resulting graph is too large and cuts the x-axis, so try increasing F to 8 to decrease the size of the circle. Try graphing the equation $x^2 + y^2 - 4x - 6y + 8 = 0$. This should be approximately correct.

c. This is an ellipse with the centre shifted down and to the left. D and E must both be positive. A must be bigger than C to make it stretch vertically. Try $A = 4$, $B = 0$, $C = 1$, $D = 5$, $E = 5$ and $F = -20$. This is about right, so the equation could be $4x^2 + y^2 + 5x + 5y - 20 = 0$.

d. This is a horizontal ellipse that has been shifted down and to the right. Because it is horizontal, A should be smaller than C, D should be negative and E should be positive. Try $A = 1$, $B = 0$, $C = 5$, $D = -7$, $E = 3$ and $F = -10$. The result is a bit too round and should be lower and more to the right. Try $C = 7$, $D = -15$, $E = 15$ and $F = 10$. Try increasing F to decrease the size. Try $F = 35$. Still too small? Try $F = 50$. The equation could be $x^2 + 7y^2 - 15x + 15y + 50 = 0$.

7. Set $B = 0$. These are all circles, so $A = C$. Each circle is centred on the origin, so $D = 0$, $E = 0$ and $F < 0$. The equation of each of these circles, then, would be $Ax^2 + Cy^2 + F = 0$. If both A and $C = 1$, then the radius squared would equal $-F$. In each case, the diameter of the circle is given. Since radius equals one-half diameter, the radii would be 6, 4, 2 and 0.5 feet respectively. The equations, or the circles from smallest to largest, would be as follows:

$$x^2 + y^2 - 0.25 = 0$$
$$x^2 + y^2 - 4 = 0$$
$$x^2 + y^2 - 16 = 0$$
$$x^2 + y^2 - 36 = 0$$

EXERCISE 3d

8. The value of $F = 0$.

9. Any value of F that is less than 8.5 will result in a circle. Without a graphing calculator, it would be safe to choose a negative value.

10. No, it means that the angle the cutting plane makes with the axis of symmetry has reached 90°. The result is still one of the four regular conic sections, but now $A = C$ so a circle results.

11. Yes, as nonzero values of D and E are added, the size of the conic is increased. In order to offset this effect, a larger F is necessary if the original size of the conic is to be maintained.

12. The circle or ellipse that degenerates to no graph is the only degenerate that cannot be demonstrated by cutting a double-napped cone or a cylinder.

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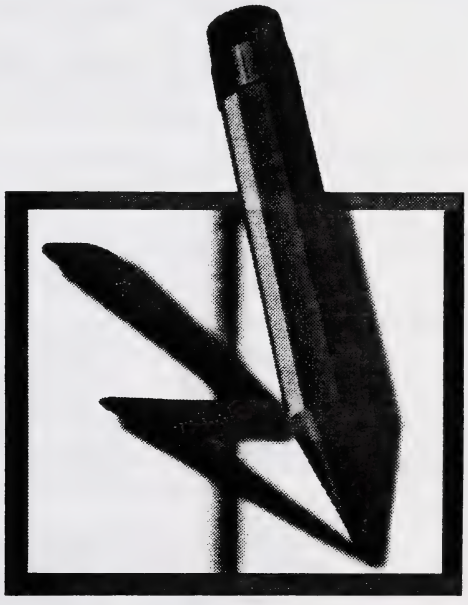
MATH FACTOR

QUADRATIC RELATIONS



Lesson 4:
General Quadratic Relation: Parabolas and Hyperbolas

Student Workbook

Produced by

THE EDUCATION STATION

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This guide accompanies **General Quadratic Relation: Parabolas and Hyperbolas** from the Math Factor series.

Writer: Roxann Trouth
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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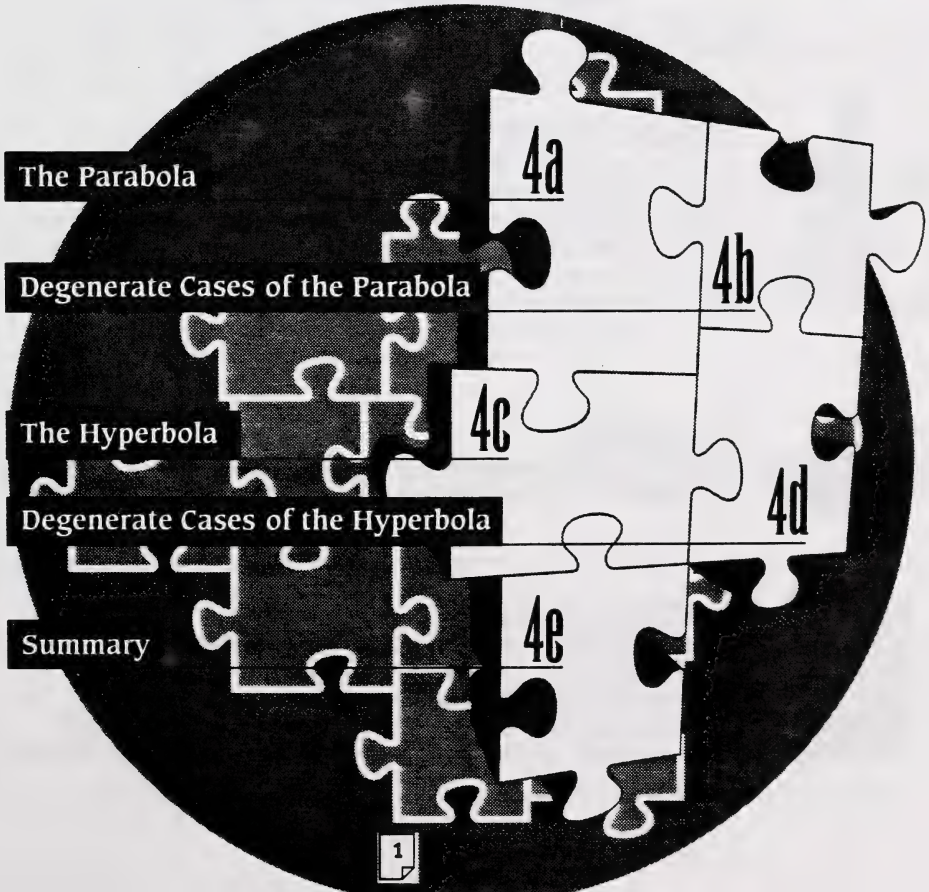
Lesson 4: GENERAL QUADRATIC RELATION: PARABOLAS AND HYPERBOLAS

LEARNER EXPECTATIONS

By the end of Lesson 4 you should be able to

- demonstrate an understanding of the general quadratic relation, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$, with respect to parabolas and hyperbolas
- describe the conics generated by various combinations of values for the numerical coefficients, in particular, parabolas and hyperbolas
- recognize that designating $B = 0$ gives an axis of symmetry parallel to one of the coordinate axes
- investigate and describe the effects of the numerical coefficients on the graphs of parabolas and hyperbolas using calculators or computers
- analyze the graphs of parabolas and hyperbolas given their equations
- investigate and describe the effects of the numerical coefficients on the orientation size and shape of the graphs of parabolas and hyperbolas

This lesson has five parts:



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Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

The Parabola

Watch the Introduction and Video Segment 4a.

Using the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$, a parabola exists when A or C is equal to 0. The values of A and C determine the orientation of the parabola and the values of D, E and F affect the width and location of the vertex on the coordinate plane.

If you do not have access to a graphing calculator or computer graphing software, the following points may help you to relate the equation of the parabola to the orientation and location of the vertex on the coordinate plane.

If A and $E \neq 0$ and $C = 0$, the result is a vertical parabola.

- If $AE < 0$, the graph opens up.
- If $AE > 0$, the graph opens down.
- If the values of A or E are changed, the graph becomes narrower or wider.
- If both $D = 0$ and $F = 0$, the vertex is on the origin.
- If $D = 0$ and $F \neq 0$, the vertex is moved vertically along the y-axis.

If $C \neq 0$, $D \neq 0$, and $A = 0$, the result is a horizontal parabola.

- If $CD < 0$, the graph opens right.
- If $CD > 0$, the graph opens left.
- If the values of C or D are changed, the graph becomes narrower or wider.
- If both $E = 0$ and $F = 0$, the vertex is on the origin.
- If $E = 0$ and $F \neq 0$, the vertex is moved horizontally along the x-axis.

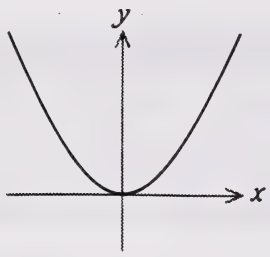
Without technology, it may be difficult to tell where the vertex will lie when there is both a horizontal and vertical shift that is when both D and $E \neq 0$.

Example

Using the previous information or a graphing program, match the following equations with the graphs.

- $2y^2 - x + 2 = 0$
- $x^2 - 2y = 0$
- $y^2 - x - 2y + 2 = 0$
- $y^2 + x = 0$
- $2x^2 + 3y - 1 = 0$

A.



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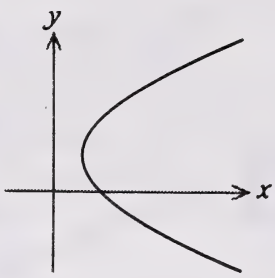

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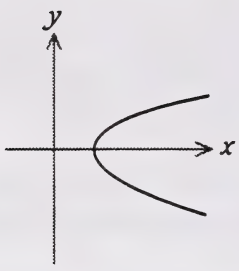
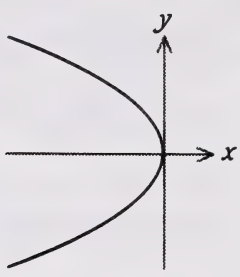


D.



Solution

The equation $2y^2 - x + 2 = 0$ has $CD < 0$; therefore, the graph opens right. $E = 0$ and $F \neq 0$; thus, the vertex is moved along the x-axis. Graph C is the match.

The equation $x^2 - 2y = 0$ has $AE < 0$; therefore, the graph opens up. $D = 0$ and $F = 0$; thus, the vertex is on the origin. Graph A is the match.

The equation $y^2 - x - 2y + 2 = 0$ has $CD < 0$; therefore, the graph opens right. $D \neq 0$, $E \neq 0$ and $F \neq 0$; thus, the vertex is moved horizontally and vertically. Graph B is the match.

The equation $y^2 + x = 0$ has $CD > 0$; therefore, the graph opens left. $D = 0$, $E = 0$ and $F = 0$; thus, the vertex is on the origin. Graph A is the match.

The equation $2x^2 + 3y - 1 = 0$ has $AE > 0$; therefore, the graph opens down. There is no match.

Exercises

1. For each of the following, determine whether the parabola is horizontal or vertical and in which quadrant the vertex can be found.

   a. $x^2 - y = 0$

   b. $x - y^2 = 0$

c. $x^2 + y - 4 = 0$

d. $x + y^2 - 4 = 0$

e. $x^2 - 4x + 5y + 10 = 0$

f. $y^2 - 4x + 5y + 10 = 0$

2. Write an equation of a horizontal parabola that opens right and with a vertex four units to the left of the origin along the x-axis. Draw a sketch of this parabola.

EXERCISE 4a SOLUTIONS are at the back of this workbook.

Degenerate Cases of the Parabola

Watch Video Segment 4b.

There are three possible degenerate cases of a parabola: a line, two parallel lines and no graph.

When $A = 0$ and $C = 0$ in the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$, the equation becomes $Dx + Ey + F = 0$, and the degenerate case of a line results. For example, the equation $3x + y - 2 = 0$ is a linear relation that may be rewritten as $y = -3x + 2$. This relation may be graphed as a straight line with slope -3 and y-intercept 2.

When $A \neq 0$, but $E = 0$ in the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$, the equation becomes $Ax^2 + Dx + F = 0$, which is a quadratic equation in one variable. The resulting degenerate case may be two parallel lines, one straight line or no graph depending on the values of A, D and F.

Example 1

The equation $x^2 - 3x - 4 = 0$ describes a degenerate case of a parabola. Describe the graph of this degenerate case.

Solution

Factor or use the quadratic formula to solve for x.

$$x^2 - 3x - 4 = 0$$
$$(x - 4)(x + 1) = 0$$
$$x - 4 = 0 \quad \text{or} \quad x + 1 = 0$$
$$x = 4 \qquad\qquad x = -1$$

The graph of these two solutions will be two parallel vertical lines with x-intercepts at 4 and -1.

Example 2

The equation $x^2 - 4x + 4 = 0$ describes a degenerate case of a parabola. Describe the graph of this degenerate case.

Solution

The equation may be factored as follows:

$$(x - 2)(x - 2) = 0$$
$$x = 2 \text{ and } x = 2$$

The graph of these solutions will be a single vertical line with the x-intercept at 2.

Example 3

The equation $x^2 - 2x + 4 = 0$ describes a degenerate case of a parabola. Describe the graph of this degenerate case.

Solution

This equation has no solution. It does not factor, and if the quadratic formula is used, then $b^2 - 4ac = -12$ and there is no real solution.

There is no graph for this equation.

Exercises

3. Each of the following equations describes a parabola. Describe how the coefficients would have to change so that each equation represents the equation of a line, a degenerate case of the parabola.

   a. $x^2 - 2y = 5$

   b. $x^2 + 10x + 15y - 20 = 0$

   c. $4y^2 + x - 3y + 20 = 0$

   d. $x - y + 3y^2 = 0$

4. Which degenerate case of the parabola results when the changes identified are made to each of the following equations?

   a. $3x^2 + 2y - 5x = 0$ (E is changed to 0.)

b. $2y^2 + 4x + 12y + 18 = 0$ (D is changed to 0.)

c. $3x^2 - x + 4y + 4 = 0$ (E is changed to 0.)

d. $5x^2 - 3x + 4y - 12 = 0$ (A is changed to 0.)

EXERCISE 4b SOLUTIONS are at the back of this workbook.

The Hyperbola

Watch Video Segment 4c.

Using the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$, a hyperbola exists when $AC < 0$. In general, as the values of A and C change, the orientation and shape of the graph are affected. Changing the values of the coefficient D shifts the graph horizontally and affects the size of the graph. Changing the values of the coefficient E shifts the centre of the graph vertically and affects the size of the graph. The coefficient F affects both placement and shape of the hyperbola.

If you do not have access to a graphing calculator or computer graphing software, the following points may help you to relate the equation of the hyperbola to the orientation and location of the vertex on the coordinate plane.

- As C becomes more negative, the graph is stretched horizontally.
- As A is increased, the graph is stretched vertically.

Before analyzing any hyperbola, ensure that A is positive. (If necessary, multiply every term in the equation by -1.)

- If $D < 0$, the centre of the graph is shifted to the right.
- If $D > 0$, the centre of the graph is shifted to the left.
- If $E < 0$, the centre of the graph is shifted down.
- If $E > 0$, the centre of the graph is shifted up.

It may be difficult to determine if a hyperbola is horizontal or vertical without the use of a graphing calculator or computer graphing program.

Exercises

5. Explain the major effect on the hyperbola $x^2 - y^2 - 10 = 0$ as the equation goes through each of the following transformations.

   a. $x^2 - y^2 + 25 = 0$

   b. $x^2 - y^2 - 25 = 0$

c. $x^2 - 5y^2 - 25 = 0$

d. $x^2 - 5y^2 + 3x - 25 = 0$

e. $x^2 - 5y^2 - 6x - 25 = 0$

f. $x^2 - 5y^2 - 2y - 25 = 0$

g. $x^2 - 5y^2 + 2y - 25 = 0$

h. $x^2 - 5y^2 + 3x + 2y - 25 = 0$

6. In order to keep the orientation of any hyperbola such that its major axes of symmetry are parallel to the x- and y-axes, which coefficient(s) must remain equal to 0?

7. In order to keep a hyperbola centred on the origin, which coefficient(s) must remain equal to 0?

EXERCISE 4c SOLUTIONS are at the back of this workbook.

Degenerate Cases of the Hyperbola

The degenerate case of a hyperbola is two intersecting straight lines. If the equation representing a hyperbola can be factored into two linear factors representing straight lines with different slopes, then the resulting graph is two intersecting lines.

Example 1

The equation $x^2 - y^2 = 0$ describes the degenerate case of the hyperbola. Algebraically show why this is true.

Solution

$$x^2 - y^2 = 0$$ Factored.

$$(x - y)(x + y) = 0$$

$$x - y = 0 \text{ or } x + y = 0$$

$$x = y \qquad x = -y$$

The solutions are the equations of two intersecting straight lines through the origin, one with slope 1 and the other with slope -1.

Example 2

The equation $x^2 - 4y^2 - 4x + 12y - 5 = 0$ describes the degenerate case of a hyperbola. Algebraically show why this is true if $x^2 - 4y^2 - 4x + 12y - 5 = (x - 2y + 1)(x + 2y - 5)$.

Solution

Set each factor equal to 0.

$$x - 2y + 1 = 0 \text{ or } x + 2y - 5 = 0$$

These equations can be rewritten in $y = mx + b$ form.

$$-2y = -x - 1 \qquad 2y = -x + 5$$

$$y = \frac{1}{2}x + \frac{1}{2} \qquad y = -\frac{1}{2}x + \frac{5}{2}$$

Both equations represent straight lines.

Because they have different slopes, they will intersect.

Note: The graphs for the previous examples can be verified using a graphing calculator or a graphing computer program.

Exercises

8. Using technology, determine which of the following equations describes a degenerate case of a quadratic relation.

   a. $x^2 - y^2 - 2x + 3y - 25 = 0$

   b. $3x^2 - 5y^2 + 3 = 0$

   c. $5x^2 - y^2 = 0$

   d. $x^2 - 4y^2 - 8x + 4y + 15 = 0$

   e. $x^2 - y^2 + 3x - 3y = 0$

   f. $-x^2 + y^2 - 5x + 2y = 0$

9. If the degenerate case of a hyperbola, two intersecting lines, intersects at a point in quadrant 3, then what can you say about the coefficients of its equation?

10. If the equation of the degenerate case of a hyperbola factors to $(x - 2y)(x + 2y + 1) = 0$, then what is the equation written in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$?

EXERCISE 4d SOLUTIONS are at the back of this workbook.

Summary

When graphed, the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$, may produce a circle, an ellipse, a hyperbola, a parabola, a point, a line, two intersecting lines, two parallel lines or no graph depending on the values of the coefficients.

The **circle** may occur when $A = C$. The degenerate case of a circle can be either a point or no graph.

The **ellipse** may occur when $A \neq 0$ and $AC > 0$. The degenerate case of an ellipse can be either a point or no graph.

The **hyperbola** may occur when $AC < 0$. The degenerate case of a hyperbola is two intersecting lines.

The **parabola** may occur if $A = 0$ or $C = 0$ (but not both). The degenerate case of a parabola can be either a line, two parallel lines or no graph.

Exercises

11. Write the equation of a circle centred at the origin. Change the equation so that its graph will still be a circle, but its centre will be in quadrant 4.

12. Write the equation of a circle centred at the origin. Change the value of F in the equation so that the equation will describe the following:

    a. the degenerate case of a point

    b. the degenerate case of no graph

13. Write the equation of a horizontal ellipse centred at the origin. Change the equation so that the ellipse becomes stretched vertically, then change the equation so that the ellipse becomes a circle.

14. Write the equation of a horizontal hyperbola centred at the origin. Change the equation so that the hyperbola becomes vertical and then change the equation so that the hyperbola becomes two straight lines.

15. Write the equation of a parabola that opens down with its vertex at the origin.

16. Write the equation of a parabola that opens to the right with its vertex at the origin. Change the equation so that it describes an ellipse.

17. Write the equation of a parabola that opens up with its vertex not at the origin. Change the equation so that it describes a straight line.

EXERCISE 4e SOLUTIONS are at the back of this workbook.

WHAT HAVE YOU LEARNED?

Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. This is the last lesson that deals with describing conic sections in terms of coefficients of the terms of the general quadratic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. In future lessons, you will learn other ways to define conic sections and to find their equations according to their locus definitions or their eccentricity.

Solutions

EXERCISE 4a

1. a. vertical parabola, opening up, vertex at the origin

   b. horizontal parabola, opening right, vertex at the origin

   c. vertical parabola, opening down, vertex moved up four units

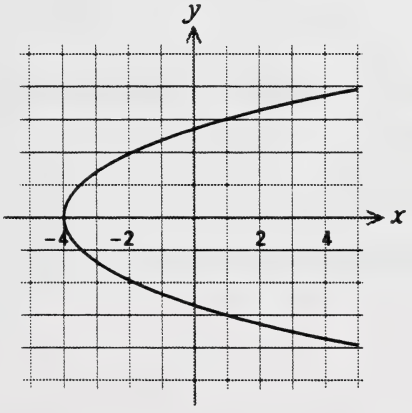
   d. horizontal parabola, opening left, vertex moved right four units

   e. vertical parabola, opening down, vertex shifted both horizontally and vertically (Vertex is in quadrant 4.)

   f. horizontal parabola, opening right, vertex shifted both horizontally and vertically (Vertex is in quadrant 4.)

2. An example might be $y^2 - x - 4 = 0$.

   If it opens right, it is horizontal; therefore, $A = 0$, $B = 0$, $C \neq 0$, $D \neq 0$. In order to open right, A and C must have different signs. If it is shifted four units left of the origin, F might be -4.



EXERCISE 4b

3. a. If $A = 0$, the equation becomes $-2y = 5$, a horizontal line with y-intercept -2.5.

   b. If $E = 0$ and $F = +25$, the equation becomes $x^2 + 10x + 25 = 0$, which factors to $(x + 5)(x + 5) = 0$, and the solution, $x = -5$, is a vertical line with x-intercept -5. Alternatively, if $A = 0$, the equation becomes $10x + 15y - 20 = 0$.

   c. If $C = 0$, the equation becomes $x - 3y + 20 = 0$.

   d. If $C = 0$, the equation becomes $x - y = 0$.

4. a. Two parallel lines. The equation becomes $3x^2 - 5x = 0$. This factors to $x(3x - 5) = 0$.

   b. One line. The equation becomes $2y^2 + 12y + 18 = 0$. This factors to $2(y + 3)(y + 3) = 0$.

   c. No graph. The equation becomes $3x^2 - x + 4 = 0$. Therefore, there is no real solution.

   d. One line. The equation becomes $-3x + 4y - 12 = 0$. Therefore, it is a linear relation.

EXERCISE 4c

5. a. The hyperbola goes from vertical to horizontal.

   b. The vertices of the hyperbola become farther apart.

   c. The graph becomes stretched along the x-axis. (The arms become closer to the x-axis.)

   d. The centre of the graph shifts to the left.

   e. The centre of the graph shifts to the right.

   f. The centre of the graph shifts down.

   g. The centre of the graph shifts up.

   h. The centre of the graph shifts left and up.

6. The coefficient B must equal 0.

7. The coefficients C and D must equal 0 and the coefficient F must not equal 0 or a degenerate occurs.

EXERCISE 4d

8. Equations C, D and E are all degenerates, two intersecting lines.

9. If $A > 0$ and $B = 0$, then $C < 0$, $D > 0$, $E < 0$ and F can be a number of different values.

10. $x^2 - 4y^2 + x - 2y = 0$

EXERCISE 4e

11. $x^2 + y^2 - 25 = 0$; $x^2 + y^2 - 5x + 3y - 10 = 0$
    Many solutions are possible. In the first equation, $A = C$, $D = 0$, $E = 0$ and $F < 0$. In the second equation, $D < 0$ and $F > 0$.

12. $2x^2 + 2y^2 - 10 = 0$; a. $2x^2 + 2y^2 = 0$; b. $2x^2 + 2y^2 + 10 = 0$
    Many solutions are possible. In the second equation, $F = 0$. In the third equation, $F > 0$.

13. $x^2 + 4y^2 - 16 = 0$; $4x^2 + y^2 - 16 = 0$; $4x^2 + 4y^2 - 16 = 0$
    In the first equation, $C > A$ and $F < 0$. In the second equation, $A > C$. In the third equation, $A = C$.

14. $x^2 - y^2 - 20 = 0$; $x^2 - y^2 + 20 = 0$; $x^2 - y^2 = 0$
    In the first equation, $AC < 0$, $D = 0$, $E = 0$ and $F < 0$. In the second equation, $F > 0$. In the third equation, $F = 0$.

15. $x^2 + y = 0$. Vertical parabola, so $A \neq 0$, $E \neq 0$ and $C = 0$. The vertex is on the origin, so $D = 0$ and $F = 0$. The parabola opens down, so A and E have the same sign.

16. $y^2 - x = 0$; $2x^2 + y^2 - x - 10 = 0$
The first equation must have C ≠ 0 and D ≠ 0, $A = 0$, $E = 0$ and $F = 0$. The second equation must include an x^2 term, $A \neq C$, and if F is negative, the ellipse becomes large enough to easily graph.

17. $2x^2 - 4x - y + 5 = 0$; $-4x - y + 5 = 0$
The first equation must have A and E with opposite signs, and D and/or $F \neq 0$. The second equation must have $A = 0$ and $C = 0$.

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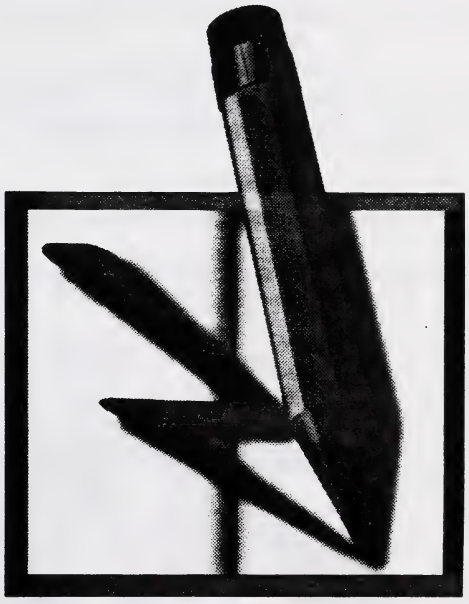
MATH FACTOR

QUADRATIC RELATIONS



Lesson 5:
Locus and Eccentricity of the Parabola

Student Workbook

Produced by

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This guide accompanies **Locus and Eccentricity of the Parabola** from the Math Factor series.

Writer: Gordon Mallett
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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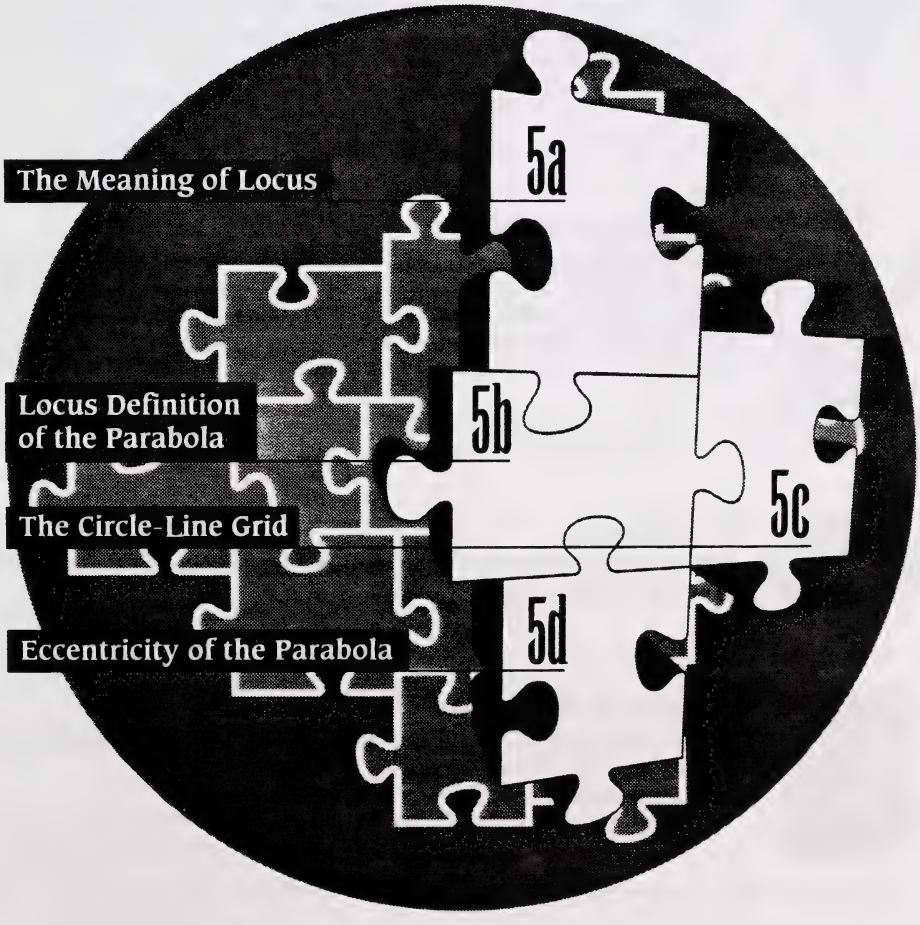
Lesson 5: LOCUS AND ECCENTRICITY OF THE PARABOLA

LEARNER EXPECTATIONS

By the end of Lesson 5 you should be able to

- recognize that there are several approaches that can be used to investigate the shape of the parabola
- solve general locus problems and parabola problems defined in terms of locus
- use a circle-line grid and a calculator or computer graphing program to generate and analyze the graphs of parabolas
- solve parabola problems that involve eccentricity

This lesson has four parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Building Blocks

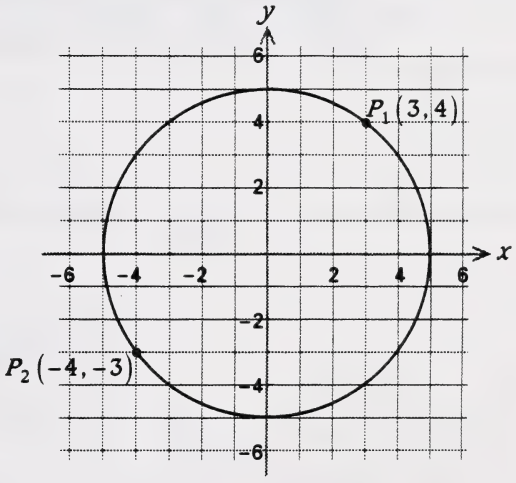
Do you remember how to use the distance formula to find the distance between any two points in a plane? Before watching this program, refresh your memory by studying the following example and doing the exercises.

Distance Formula

The distance between the two points, (x_1, y_1) and (x_2, y_2), in a plane is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example

Use the distance formula to find the distance between points P_1 and P_2 in the circle at the right.



Solution

$P_1 = (x_1, y_1) = (3, 4)$, and
$P_2 = (x_2, y_2) = (-4, -3)$.

$$\therefore d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-4-3)^2 + (-3-4)^2}$$
$$= \sqrt{(-7)^2 + (-7)^2}$$
$$= \sqrt{49 + 49}$$
$$\doteq 9.9$$

The distance is about 9.9 units.

Exercises

1. Determine the exact value of the distance between the points $(9, 2)$ and $(6, 8)$.

2. Determine the length of the diameter of the circle in the example.

BUILDING BLOCKS SOLUTIONS are at the back of this workbook.

The Meaning of Locus

The **locus** is a set of points, lines or curves that satisfy one or more given conditions.

Challenge yourself

Find the equation of the locus of points that are equidistant from two fixed points $(1, 5)$ and $(5, 3)$.

MATH FACTOR

Watch the Introduction and Video Segment 5a.

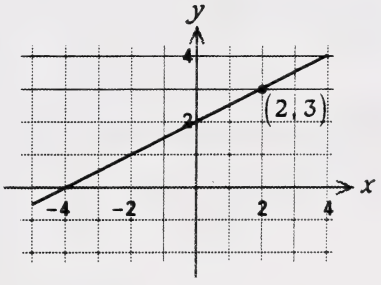
Example

A line contains the point $(2, 3)$ and has a slope of $\frac{1}{2}$. Find the equation of the locus; sketch and describe the locus.

Solution

Substitute into the slope-intercept formula.

$$y = mx + b$$
$$3 = \frac{1}{2}(2) + b$$
$$3 = 1 + b$$
$$b = 2$$
$$\therefore y = \frac{1}{2}x + 2$$



The locus is all the points that lie on the line.

TEXT

Additional Examples
Mathematics 12: p. 86 (Examples 1, 2)
Mathematics: Principles and Process 12: pp. 298-299
Holtmath 12: p. 150 (Example 1)

Exercises

1. Determine the equation of the locus of points that are equidistant from the points $(-5,5)$ and $(5,-5)$.

2. A locus contains the points $(9,2)$ and $(x,8)$. If these points are $3\sqrt{5}$ units apart, then determine the possible value or values of x.

EXERCISE 5a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 88 (Questions 1, 3)
Mathematics: Principles and Process 12: p. 300 (Questions 3, 5)
Holtmath 12: p. 152 (Question 5)

Locus Definition of the Parabola

Challenge yourself

A parabola has a focus $F(2, 3)$ and a directrix $x = 10$. Determine the equation of this parabola.

MATH FACTOR

Watch Video Segment 5b.

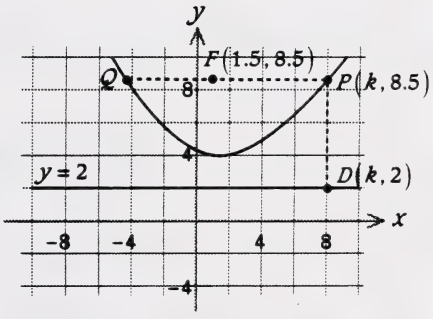
The parabola, like other quadratic relations, can be defined in terms of locus. A **parabola** is the locus of all the points that are equidistant from a fixed point and a fixed line.

Note: The fixed point is referred to as the **focus**, and the fixed line is referred to as the **directrix**. In formula form, this parabola definition is written as $|\overline{PF}| = |\overline{PD}|$.

Example

The locus at the right is a parabola with a focus $F(1.5, 8.5)$ and a directrix $y = 2$.

Determine the value of k and explain what it means.



Solution

$$|\overline{PF}| = |\overline{PD}|$$

$$\sqrt{(k-1.5)^2 + (8.5-8.5)^2} = \sqrt{(k-k)^2 + (8.5-2)^2}$$

$$k^2 - 3k + 2.25 = (6.5)^2$$

$$k^2 - 3k + 2.25 = 42.25$$

$$k^2 - 3k - 40 = 0$$

$$(k-8)(k+5) = 0$$

$$k = 8 \text{ or } k = -5$$

Because the diagram indicates that k is positive, $k = 8$. The value of $k = -5$ indicates that another point, $Q(-5, 8.5)$, is on the left arm of the parabola.

Note: The value of k can also be obtained by inspecting the graph:

$$|\overline{PD}| = 6.5$$

So, $|\overline{PF}| = 6.5$ and k must be 8.

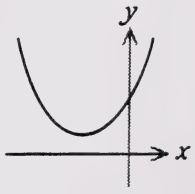
Additional Examples
Mathematics 12: p. 87 (Example 3)
Mathematics: Principles and Process 12: p. 312
Holtmath 12: p. 160

Exercises

3. The focus of a parabola is $(4, -2)$, and the directrix is $y + 6 = 0$. Determine the equation and sketch the locus.

4. Point $P(3, 4)$ is on the graph of a parabola and is 10.6 units from the focus. How far is the point P from the directrix?

5. To determine whether the graph shown is a parabola or one branch of a hyperbola, which of the following information is needed.



- the coordinates of the focus
- that $x = -1$ is an axis of symmetry
- the domain
- the range
- the eccentricity
- the coordinates of a point on the graph

EXERCISE 5b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 88 (Question 6)
Mathematics: Principles and Process 12: p. 313 (Questions 1, 7)
Holtmath 12: p. 162 (Questions 3, 5)

The Circle-Line Grid

challenge yourself

To help sketch the locus of points that describe parabolas, a special type of grid, called the circle-line grid, can be used. Its use is explained in Video Segment 5c.

MATH FACTOR

Watch Video Segment 5c.

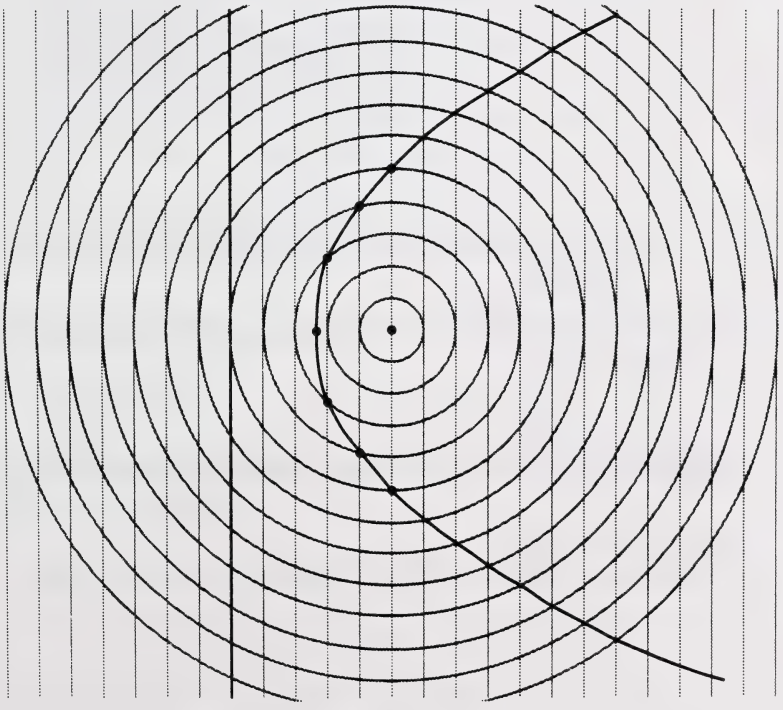
Example

The distance from the focus to the directrix of a parabola is five units. Draw a sketch of the locus on a circle-line grid by plotting the vertex and six other points.

Solution

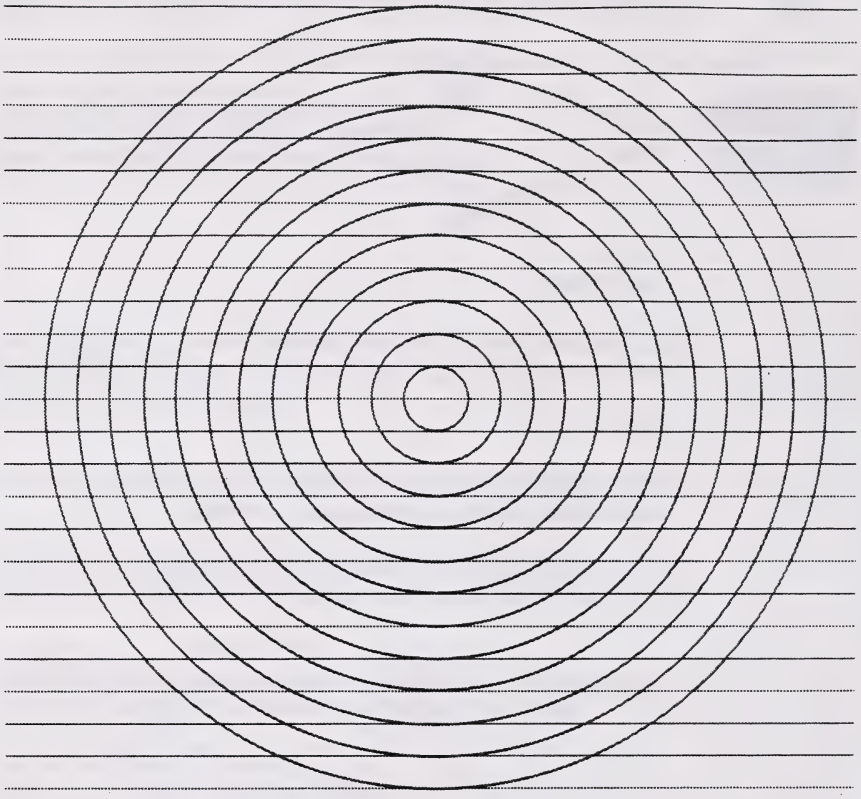
In this solution, the grid is oriented so that the lines run vertically. The line five units to the left of the focus is chosen as the directrix.

The vertex is positioned halfway (about 2.5 units) between the focus and directrix. The other plotted points are equidistant from the focus and directrix.



Exercises

6. The vertex of a parabola is positioned four units above the focus. Indicate the locus by drawing the directrix and then plotting the vertex and several other points on the following circle-line grid.



7. a. On the grid in question 6, show the locus for which the focus to directrix distance is six units.

   b. Compare its shape to the shape of the parabola in question 6.

EXERCISE 5c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics: Principles and Process 12: p. 311 (Questions 1, 2)

Eccentricity of the Parabola

The ratio of $|\overline{PF}|$ to $|\overline{PD}|$ is referred to as the eccentricity, e, of a conic section.

$$\text{Eccentricity}(e) = \frac{|\overline{PF}|}{|\overline{PD}|}$$

Conic	Eccentricity
ellipse	$0 < e < 1$
parabola	$e = 1$
hyperbola	$e > 1$

MATH FACTOR

Watch Video Segment 5d.

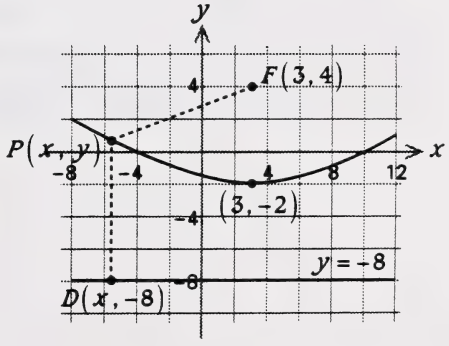
Example

The focus of a parabola is $F(3, 4)$ and its vertex is $(3, -2)$. Using this information, do the following:

- Find the directrix.
- Sketch the graph of the parabola.
- Find the domain and range.
- Determine the equation of the locus.

Solution

The focus-vertex distance is 6; thus, the vertex-directrix distance is also 6. The directrix must be the line $y = -8$.



The domain and range are as follows:

- Domain: $x \in R$
- Range: $y \geq -2$

To determine the equation, choose $P(x, y)$ on the parabola.

$$|\overline{PF}| = |\overline{PD}|$$

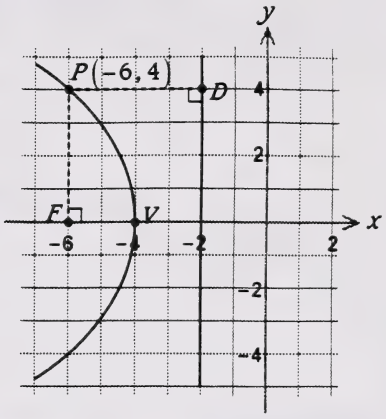
$$\sqrt{(x-3)^2 + (y-4)^2} = \sqrt{(x-x)^2 + (y+8)^2}$$

$$x^2 - 6x + 9 + y^2 - 8y + 16 = y^2 + 16y + 64$$

$$x^2 - 6x - 24y - 39 = 0$$

Exercises

8. The graph of the quadratic relation at the right has a focus $F(-6, 0)$ and directrix $x = -2$.



   a. Verify that this is the graph of a parabola.

   b. Find the coordinates of point V.

   c. State the domain and range of the relation.

9. The eccentricity of a quadratic relation is 1. Which of the following equations could represent this relation? Explain your answer.

   A. $y^2 + 8x - 6y - 23 = 0$

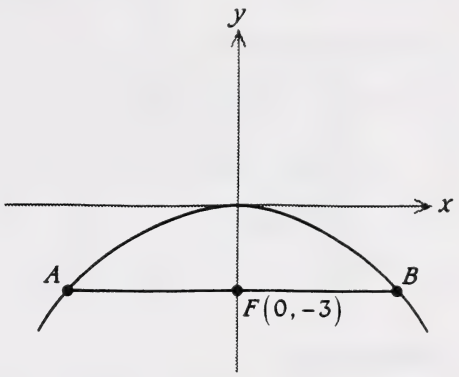
   B. $16x^2 - 9y^2 - 144 = 0$

   C. $16x^2 + 9y^2 - 144 = 0$

   D. $x^2 + y^2 - 100 = 0$

10. The parabola shown has a focus $F(0,-3)$ and a vertex $(0,0)$.

a. Find the equation of the directrix.



b. Find the equation of the locus.

c. Find the length of $\left|\overline{AB}\right|$.

EXERCISE 5d SOLUTIONS are at the back of this workbook.

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of the lesson and check off the points that you have mastered. Go back to the video and study those parts that you find difficult. Knowledge of locus definitions and eccentricity is required in **Lesson 6: Eccentricity of the Ellipse**.

Solutions

BUILDING BLOCKS

1. $d = \sqrt{(6-9)^2 + (8-2)^2}$

$= \sqrt{(-3)^2 + (6)^2}$

$= \sqrt{9+36}$

$= \sqrt{45}$

$= 3\sqrt{5}$ units

2. Let $y = 0$.

$\therefore x^2 + 0^2 - 25 = 0$

$x^2 = 25$

$x = \pm 5$

The diameter on the x-axis stretches from -5 to 5. So, the diameter is 10 units.

EXERCISE 5a

1. $\sqrt{[x-(-5)]^2 + (y-5)^2} = \sqrt{(x-5)^2 + [y-(-5)]^2}$

$(x+5)^2 + (y-5)^2 = (x-5)^2 + (y+5)^2$

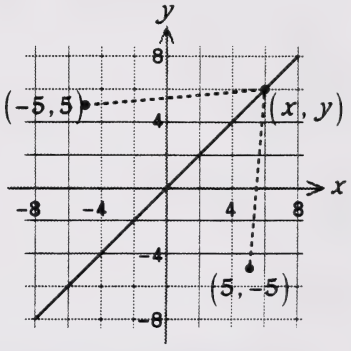
$x^2 + 10x + 25 + y^2 - 10y + 25 = x^2 - 10x + 25 + y^2 + 10y + 25$

$10x - 10y = -10x + 10y$

$-20y = -20x$

$y = x$

The equation of the locus is the line $y = x$.



2. Apply the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

$3\sqrt{5} = \sqrt{(x-9)^2 + (8-2)^2}$

$(3\sqrt{5})^2 = \left[\sqrt{(x-9)^2 + (8-2)^2}\right]$

$9(5) = (x-9)^2 + 6^2$

$45 = x^2 - 18x + 81 + 36$

$0 = x^2 - 18x + 72$

$0 = (x-12)(x-6)$

$x = 12$ or $x = 6$

Two x-coordinates are possible: 6 and 12.

EXERCISE 5b

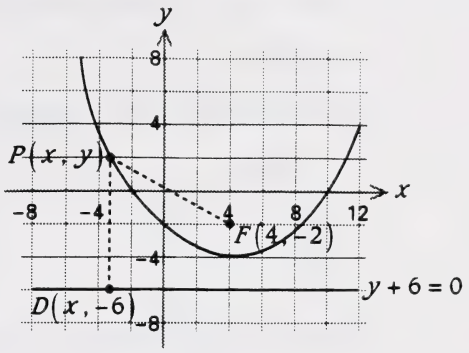
3. $|\overline{PF}| = |\overline{PD}|$

$$\sqrt{(x-4)^2+[y-(-2)]^2} = \sqrt{(x-x)^2+[y-(-6)]^2}$$

$$(x-4)^2+(y+2)^2 = (y+6)^2$$

$$x^2-8x+16+y^2+4y+4 = y^2+12y+36$$

$$x^2-8x-8y-16 = 0$$



4. $|\overline{PF}| = |\overline{PD}|$

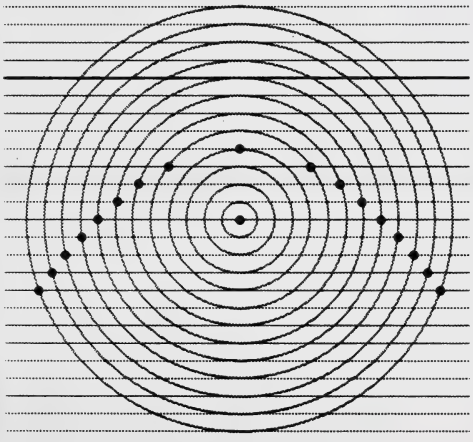
$|\overline{PF}| = 10.6$ units

$\therefore |\overline{PD}| = 10.6$ units

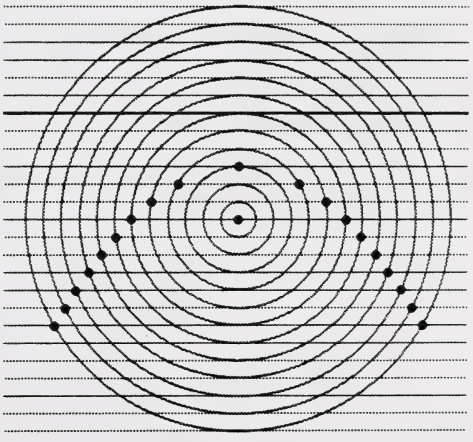
5. To determine whether the graph is a parabola or one branch of a hyperbola, you must know the eccentricity or the coordinates of a point on the graph.

EXERCISE 5c

6.



7. a.



b. The parabola in question 7 is narrower than the parabola in question 6.

EXERCISE 5d

8. a. P and F have the same x-coordinates, so the vertical distance between them is four units.

$$\left|\overline{PF}\right| = 4$$

P and D have the same y-coordinates, so the horizontal distance between them is four units.

$$\left|\overline{PD}\right| = 4$$

$$\therefore \frac{\left|\overline{PF}\right|}{\left|\overline{PD}\right|} = \frac{4}{4} = 1$$

Therefore, this is a graph of a parabola.

b. The vertex , V, of a parabola falls halfway between the focus and directrix.
The focus to directrix distance is four units, so the vertex has an x-coordinate of -4.

$$V(-4, 0)$$

c. Domain: $x \leq -4$
Range: $y \in R$

9. $\dfrac{\left|\overline{PF}\right|}{\left|\overline{PD}\right|} = 1$

Therefore, the relation is a parabola.
$AC = 0$ for a parabola.
In A, $AC = 0 \times 1 = 0$.
Therefore, the answer is A.

10. a. The vertex to focus distance is three units. The vertex to directrix distance is also three units. So, the directrix is the line $y = 3$.

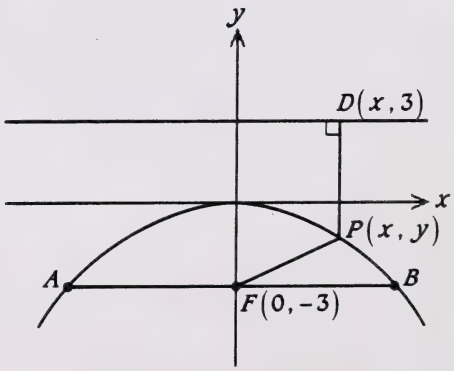
b. Choose $P(x, y)$ on the parabola.

$$\left|\overline{PF}\right| = \left|\overline{PD}\right|$$

$$\sqrt{(x-0)^2 + (y+3)^2} = \sqrt{(x-x)^2 + (y-3)^2}$$

$$x^2 + y^2 + 6y + 9 = y^2 - 6y + 9$$

$$x^2 + 12y = 0$$



c. Points A and B are on the parabola, so they must satisfy the equation.

At points A and B, $y = -3$.

$$\therefore x^2 + 12(-3) = 0$$
$$x^2 - 36 = 0$$
$$x^2 = 36$$
$$x = \pm 6$$

The points are $A(-6, 3)$ and $B(6, 3)$.

The length of $|\overline{AB}|$ is 12 units.

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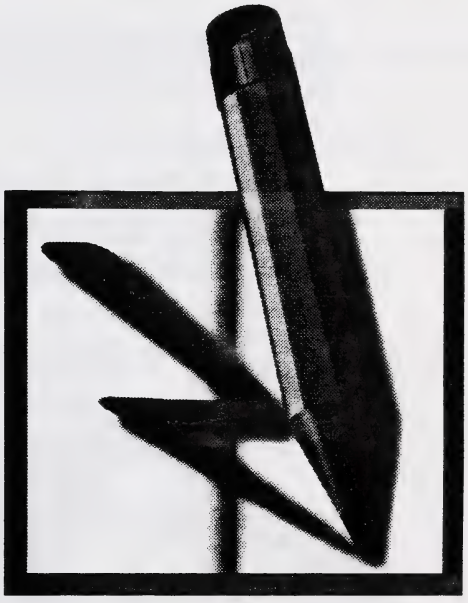
MATH FACTOR

QUADRATIC RELATIONS



Lesson 6:
Eccentricity of the Ellipse

Student Workbook

Produced by

THE EDUCATION STATION

for

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This guide accompanies **Eccentricity of the Ellipse** from the Math Factor series.

Writer: Gordon Mallett
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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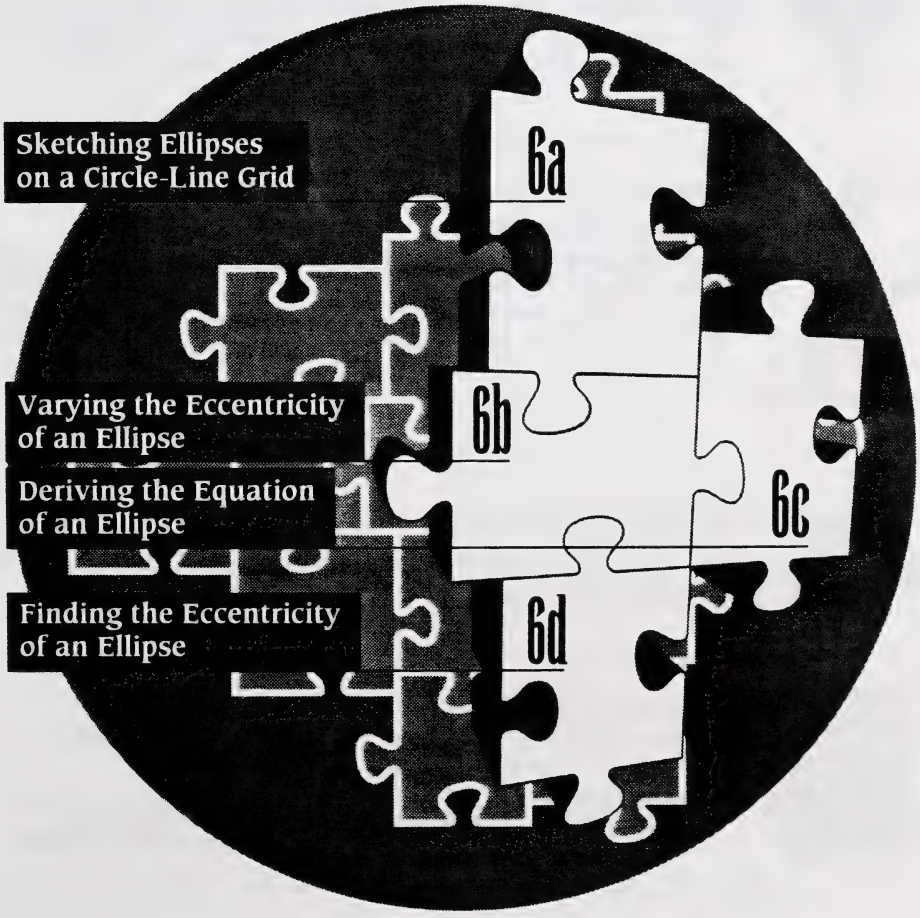
Lesson 6: ECCENTRICITY OF THE ELLIPSE

LEARNER EXPECTATIONS

By the end of Lesson 6 you should be able to:

- use a circle-line grid, and calculators or computer graphing programs, to generate and analyze the graphs of ellipses
- explain the changes in the shape of an ellipse as the eccentricity varies between the values of 0 and 1
- define an ellipse and verify its equation on the basis of its eccentricity
- derive and verify the equation of an ellipse given its eccentricity, focus and directrix
- find the eccentricity of an ellipse given its focus, directrix and point(s) on the locus

This lesson has four parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Sketching Ellipses on a Circle-Line Grid

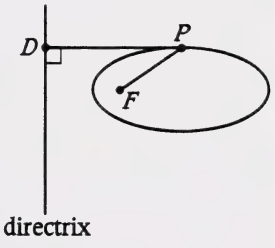
To help sketch the locus of points that describe ellipses the circle-line grid can be used. Its use is explained in Video Segment 6a.

Watch the Introduction and Video Segment 6a.

Definition:

An ellipse is the locus of a point, P, that moves so the ratio of its distance from a fixed point, F, (the focus) to its distance from a fixed line (the directrix) has a value between 0 and 1.

$$0 < \frac{|\overline{PF}|}{|\overline{PD}|} < 1$$



Example

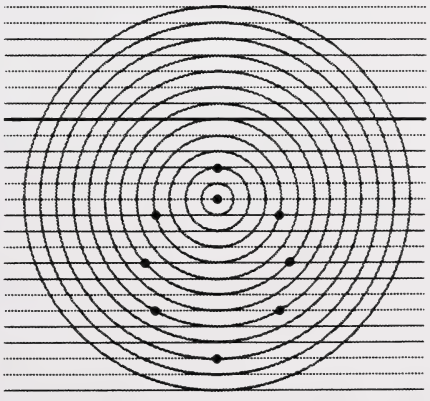
The distance from the focus to the directrix of an ellipse is five units. If $\frac{|\overline{PF}|}{|\overline{PD}|} = \frac{2}{3}$, indicate the locus by drawing the directrix and several points on a circle-line grid.

Solution

$$\frac{|\overline{PF}|}{|\overline{PD}|} = \frac{2}{3}$$

$$\therefore |\overline{PF}| = \frac{2}{3}|\overline{PD}|$$

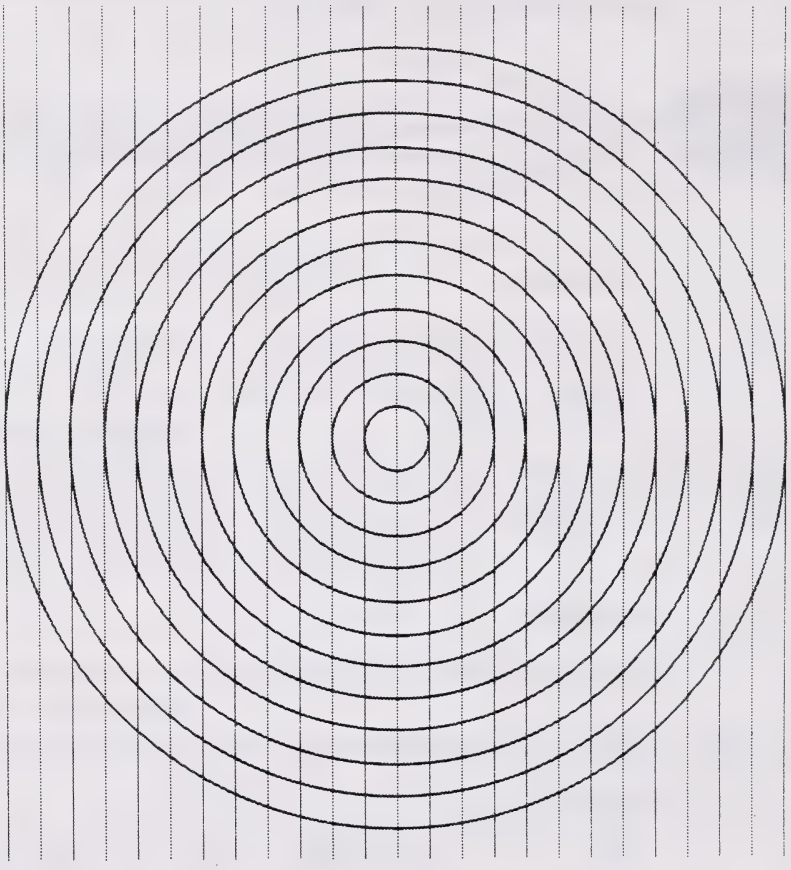
$\lvert\overline{PF}\rvert$	$\lvert\overline{PD}\rvert$
2	3
4	6
6	9
8	12
10	15



Note: In this solution, the grid is oriented so that the lines run horizontally. The line five units above the focus is taken as the directrix.

Exercises

1. The directrix of an ellipse is located four units to the left of the focus. If $\frac{|\overline{PF}|}{|\overline{PD}|} = \frac{3}{4}$, indicate the locus by drawing the directrix and several points on the following circle-line grid. Join the points with a smooth curve.



2. Using the grid in question 1, draw the ellipse with an eccentricity of $\frac{1}{2}$. Compare the shapes of the two ellipses and describe what you notice.

3. Using the grid in question 1, draw the locus in which $|\overline{PF}| = |\overline{PD}|$ and identify the type of conic formed.

EXERCISE 6a SOLUTIONS are at the back of this workbook.

Varying the Eccentricity of an Ellipse

The ratio of $|\overline{PF}|$ to $|\overline{PD}|$ is referred to as the eccentricity, e, of a conic section. For the ellipse, $0 < e < 1$.

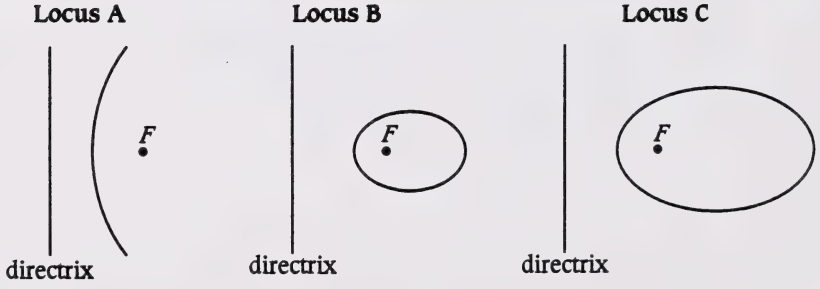
$$\text{Eccentricity}(e) = \frac{|\overline{PF}|}{|\overline{PD}|}$$

Conic	Eccentricity
ellipse	$0 < e < 1$
parabola	$e = 1$
hyperbola	$e > 1$

Example

The focus and directrix are five units apart. Match each of the following eccentricities with the sketch of its locus. Give a reason for your answer.

- $e = \frac{1}{3}$
- $e = \frac{2}{3}$
- $e = \frac{3}{3}$



Solution

$e = \frac{1}{3}$ matches with Locus B. For small values of e, the ellipse appears to be nearly circular.

$e = \frac{2}{3}$ matches with Locus C. As e increases, the ellipse gets longer and narrower.

$e = \frac{3}{3}$ matches with Locus A. $e = 1$ gives a parabola.

Exercises

4. The focus of an ellipse is five units from the directrix, and its eccentricity is 0.7.

   a. What happens to the ellipse if the focus is moved farther from the directrix while the eccentricity remains the same? Describe how this relates to the physical model of a plane intersecting a cone.

   b. What happens to the ellipse when the directrix is moved closer to the focus while the eccentricity remains the same? Describe how this relates to the physical model of a plane intersecting a cone.

   c. What happens to the ellipse when the distance between the focus and directrix remains the same but the eccentricity changes? Describe how this relates to the physical model of a plane intersecting a cone.

5. Describe what happens to the shape of an ellipse if both its eccentricity and the distance between the focus and the directrix increases.

EXERCISE 6b SOLUTIONS are at the back of this workbook.

Deriving the Equation of an Ellipse

Challenge yourself

Derive the equation of an ellipse with focus $F(-2, 3)$, directrix $y = 5$ and $e = 0.5$.

MATH FACTOR

Watch Video Segment 6c.

Use the distance formula to find the distance between any two points in a plane.

Distance Formula

The distance between two points, (x_1, y_1) and (x_2, y_2), in a plane is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example

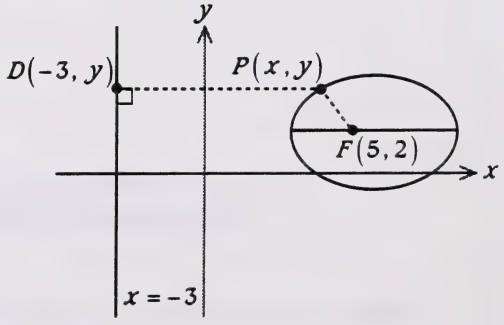
Sketch and determine the equation of the locus of points that are one-third as far from the point $(5, 2)$ as they are from the line $x = -3$. How could the equation be verified?

Note: The point is called the focus and the line is called the directrix. Because the eccentricity is between 0 and 1, the locus is an ellipse. To find the equation, identify any point, P, on the ellipse and use the distance formula.

Solution

$$|\overline{PF}| = \frac{1}{3}|\overline{PD}| \quad \text{or} \quad \frac{|\overline{PF}|}{|\overline{PD}|} = \frac{1}{3}$$

$$\frac{\sqrt{(x-5)^2 + (y-2)^2}}{\sqrt{[x-(-3)]^2 + (y-y)^2}} = \frac{1}{3}$$



Square both sides; then simplify to the general form.

$$\frac{(x-5)^2 + (y-2)^2}{(x+3)^2} = \frac{1}{9}$$

$$9\left[(x-5)^2 + (y-2)^2\right] = (x+3)^2$$

$$9x^2 - 90x + 9y^2 - 36y + 261 = x^2 + 6x + 9$$

$$8x^2 + 9y^2 - 96x - 36y + 252 = 0$$

The result can be verified in the following way.

Note that $AC > 0$; therefore, this equation could describe an ellipse. A graphing calculator could help to show that the graph is similar to the preceding one.

To show algebraically that the answer is correct, use the coordinates of any point on the ellipse (e.g., the vertex) to show that $\frac{|\overline{PF}|}{|\overline{PD}|} = \frac{1}{3}$.

Exercises

6. Using the previous example, find the coordinates of the vertices; then verify that they satisfy the condition $\frac{|\overline{PF}|}{|\overline{PD}|} = \frac{1}{3}$.

7. Find the equation of the locus of points that are one-half as far from the point $P(-4, 2)$ as they are from the line $x = -5$.

8. What information in question 7 indicates that the locus is an ellipse?

9. What information can be used from the equation derived in question 7 to identify the locus as an ellipse?

EXERCISE 6c SOLUTIONS are at the back of this workbook.

Finding the Eccentricity of an Ellipse

Challenge yourself

A point on the graph of an ellipse is $P(-6, 4)$. The focus of the ellipse is $F(-3, 0)$ and its corresponding directrix is $x = 1$. Correct to the nearest hundredth, determine the eccentricity of this ellipse.

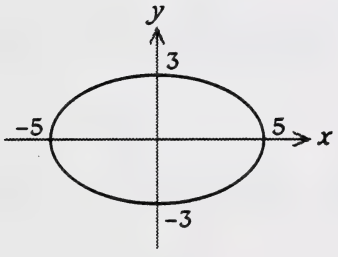
MATH FACTOR

Watch Video Segment 6d.

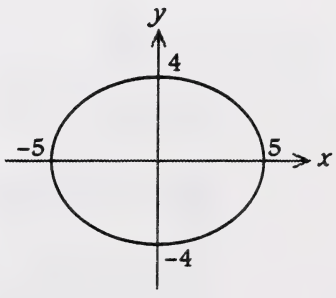
Exercises

10. The eccentricities of Ellipse 1 and Ellipse 2 are e_1 and e_2 respectively.

Ellipse 1



Ellipse 2



Which statement correctly relates the eccentricities of the ellipses? Justify your answer.

A. $1 < e_1 < e_2$
B. $1 < e_2 < e_1$
C. $0 < e_1 < e_2 < 1$
D. $0 < e_2 < e_1 < 1$

11. The line $y = -5$ is the directrix, and the point $(-3, 7)$ is the focus of a conic. The point $(-3, 3)$ is a point on the locus. Determine the eccentricity of this conic.

12. Use your answer to question 11 to determine which equation best represents the conic.

A. $4x^2 + 3y^2 + 24x - 66y + 207 = 0$

B. $x^2 - 8x + 10y - 11 = 0$

C. $x^2 + y^2 - 4x + 2y - 11 = 0$

D. $16x^2 - 9y^2 - 144 = 0$

13. Show a method for verifying that the equation you selected in question 12 is the equation that describes the conic in question 11.

EXERCISE 6d SOLUTIONS are at the back of this workbook.

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of the lesson and check off the points that you have mastered. Go back to the video and study those parts that you find difficult. Knowledge of locus definitions and eccentricity is required in **Lesson 7: Alternate Locus Definitions: Ellipse and Circle**.

Solutions

EXERCISE 6a

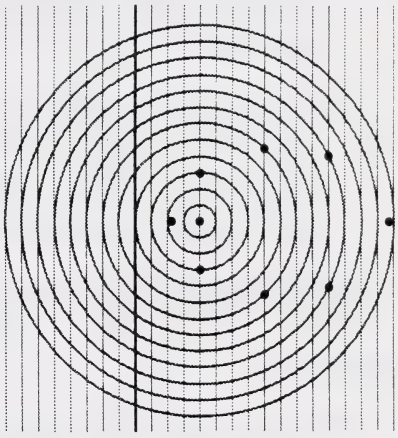
1. $\dfrac{|\overline{PF}|}{|\overline{PD}|} = \dfrac{3}{4}$

$|\overline{PF}| = \dfrac{3}{4}|\overline{PD}|$

$\lvert\overline{PF}\rvert$	$\lvert\overline{PD}\rvert$
3	4
6	8
9	12
12	16

Note: To find the left vertex, let it be u units from the focus.

$$\frac{u}{(4-u)} = \frac{3}{4}$$
$$4u = 12 - 3u$$
$$7u = 12$$
$$u = 1\frac{5}{7}$$
$$\doteq 1.7$$



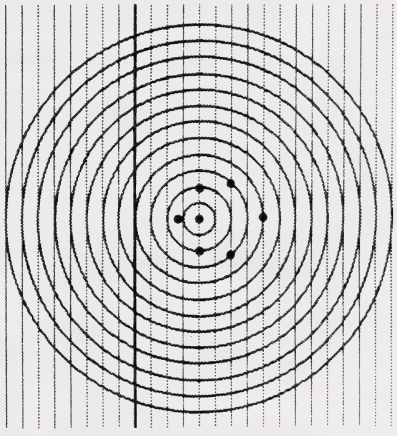
2. $\dfrac{|\overline{PF}|}{|\overline{PD}|} = \dfrac{1}{2}$

$|\overline{PF}| = \dfrac{1}{2}|\overline{PD}|$

$\lvert\overline{PF}\rvert$	$\lvert\overline{PD}\rvert$
2	4
3	6
4	8

Note: To find the left vertex, let it be u units from the focus.

$$\frac{u}{(4-u)} = \frac{1}{2}$$
$$4 - u = 2u$$
$$3u = 4$$
$$u = \frac{4}{3}$$
$$\doteq 1.3$$

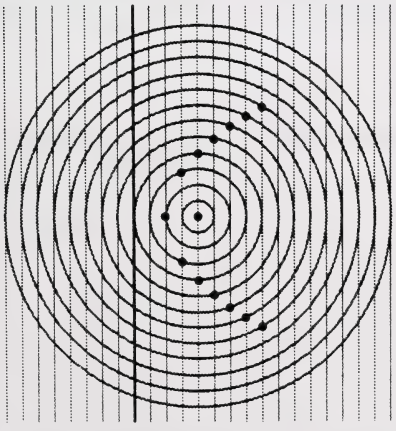


3. $|\overline{PF}| = |\overline{PD}|$

$$\therefore \frac{|\overline{PF}|}{|\overline{PD}|} = 1$$

The conic is a parabola.

$\lvert\overline{PF}\rvert$	$\lvert\overline{PD}\rvert$
2	2
3	3
4	4
5	5
6	6
7	7
8	8



EXERCISE 6b

4. a. The ellipse becomes larger as the distance between the focus and the directrix increases, but its shape generally remains the same. In the physical model, the plane intersecting the cone moves farther from the vertex, but the angle of inclination of the plane remains the same.

   b. The ellipse becomes smaller, but the shape remains the same. In the physical model, the plane intersecting the cone moves closer to the vertex, but the angle of inclination of the plane remains the same.

   c. The closer the eccentricity is to 1, the larger and more elongated the ellipse. The closer the eccentricity is to 0, the smaller and more circular the ellipse.

   In the physical model, the closer the eccentricity is to 0, the closer the plane gets to intersecting the cone at an angle of 90° to the vertical axis. The closer the eccentricity is to 1, the closer the plane gets to cutting the vertical axis at the angle between the generator and the base.

5. The ellipse gets longer and narrower because the eccentricity increases. It increases in size because the focus-directrix distance increases.

EXERCISE 6c

6. The y-coordinate of each vertex is 2.

$$\begin{aligned} 8x^2 + 9y^2 - 96x - 36y + 252 &= 0 \\ 8x^2 + 9(2)^2 - 96x - 36(2) + 252 &= 0 \\ 8x^2 - 96x + 216 &= 0 \\ x^2 - 12x + 27 &= 0 \\ (x-3)(x-9) &= 0 \end{aligned}$$

Therefore, the vertices are $(3, 2)$ and $(9, 2)$.

For vertex $(3, 2)$, $|\overline{PF}| = 2$ and $|\overline{PD}| = 6$. Thus, $\dfrac{|\overline{PF}|}{|\overline{PD}|} = \frac{1}{3}$.

For vertex $(9, 2)$, $|\overline{PF}| = 4$ and $|\overline{PD}| = 12$. Thus, $\dfrac{|\overline{PF}|}{|\overline{PD}|} = \frac{1}{3}$.

7. $\dfrac{|\overline{PF}|}{|\overline{PD}|} = \dfrac{1}{2}$

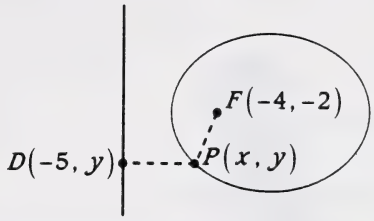
$$\frac{\sqrt{[x-(-4)]^2+[y-(-2)]^2}}{\sqrt{[x-(-5)]^2+(y-y)^2}} = \frac{1}{2}$$

$$\frac{(x+4)^2+(y+2)^2}{(x+5)^2} = \frac{1}{4}$$

$$4\left[(x+4)^2+(y+2)^2\right] = (x+5)^2$$

$$4x^2+32x+4y^2+16y+80 = x^2+10x+25$$

$$3x^2+4y^2+22x+16y+55 = 0$$



8. $|\overline{PF}|$ is given as $\frac{1}{2}$ of $|\overline{PD}|$.

$\therefore |\overline{PF}| = \frac{1}{2}|\overline{PD}|$

$$\frac{|\overline{PF}|}{|\overline{PD}|} = \frac{1}{2}$$

Thus, $e = \frac{1}{2}$ and the locus is an ellipse.

9. $A = 3$ and $C = 4$

$AC = 3 \times 4$
$= 12$

$AC > 0$ and $A \neq C$

Therefore, the locus is an ellipse.

EXERCISE 6d

10. The eccentricity of an ellipse is between 0 and 1. So, A and B are not possible. Ellipse 1 is less circular than Ellipse 2. So, its eccentricity, e_1, is larger than e_2.

Therefore, D ($0 < e_2 < e_1 < 1$) is correct.

11. $|\overline{PF}| = |3-7|$ $\quad$ $|\overline{PD}| = |3-(-5)|$

$= 4$ $\quad$ $= 8$

$$\therefore \frac{|\overline{PF}|}{|\overline{PD}|} = \frac{4}{8}$$

$$= \frac{1}{2}$$

Thus, the eccentricity is $\frac{1}{2}$.

12. If $e = \frac{1}{2}$, the conic is an ellipse.

Option A is an ellipse ($AC > 0, A \neq C$); option B is a parabola ($C = 0$); option C is a circle ($A = C$); and option D is a hyperbola ($AC < 0$).

Therefore, A is the correct response.

13. To verify that the equation in option A from question 12 is correct, let (x, y) be a point on the locus, and use the eccentricity definition to obtain the equation of the conic.

$$\frac{|\overline{PF}|}{|\overline{PD}|} = \frac{1}{2}$$

$$\frac{\sqrt{(x+3)^2 + (y-7)^2}}{y-(-5)} = \frac{1}{2}$$

$$2\sqrt{x^2+6x+9+y^2-14y+49} = y+5$$

$$4\left(x^2+6x+9+y^2-14y+49\right) = y^2+10y+25$$

$$4x^2+3y^2+24x-66y+207 = 0$$

This is the same equation as given in option A in question 12.

C.111-311c2 07

MATH FACTOR

QUADRATIC RELATIONS

Lesson 7:
Alternate Locus Definitions: Ellipse and Circle

Student Workbook

Produced by

THE EDUCATION STATION

for

Alberta EDUCATION

This guide accompanies **Alternate Locus Definitions: Ellipse and Circle** from the Math Factor series.

Writer: Gordon Mallett
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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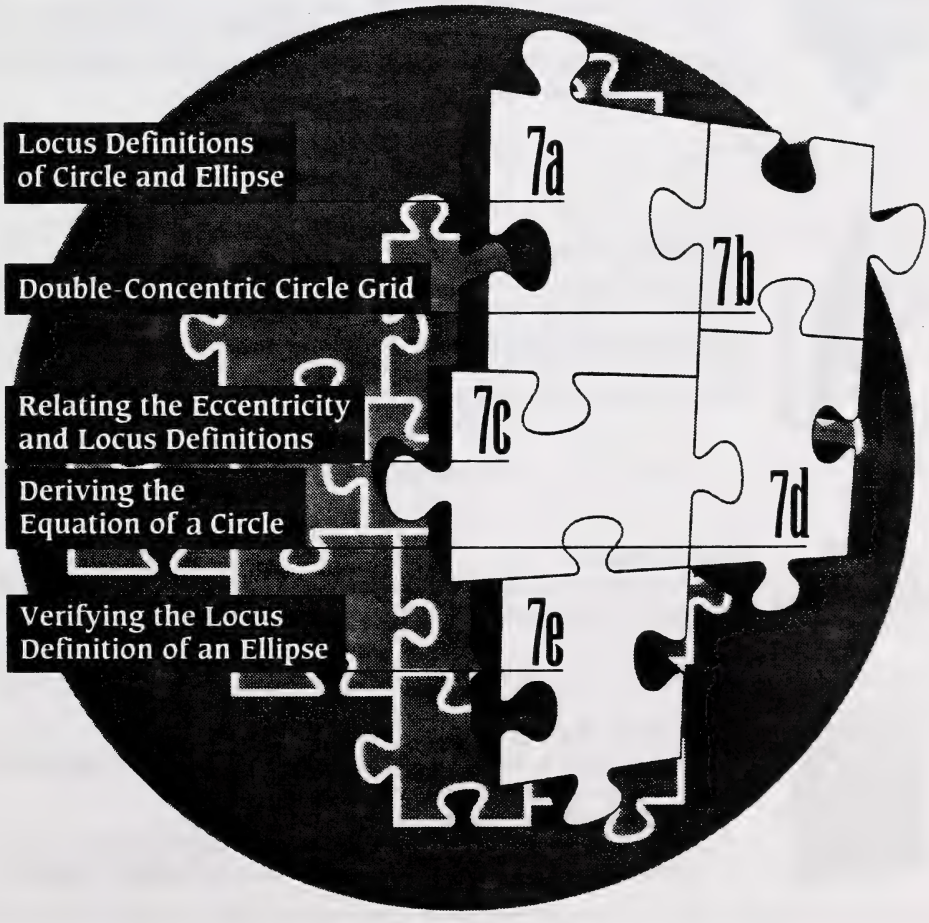
Lesson 7: ALTERNATE LOCUS DEFINITIONS: ELLIPSE AND CIRCLE

LEARNER EXPECTATIONS

By the end of Lesson 7 you should be able to

- define and use the locus definitions for the circle and ellipse
- use a double-concentric circle grid to sketch an ellipse
- show how the eccentricity and locus definitions relate to one another
- verify the locus definition of an ellipse given its equation and the two foci

This lesson has five parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Locus Definitions of Circle and Ellipse

Of all the conic sections, only the ellipse and hyperbola can be defined in terms of a locus involving two fixed points (two foci). The locus definition of a circle involves only one fixed point (the centre).

MATH FACTOR

Watch the Introduction and Video Segment 7a.

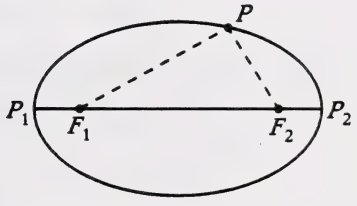
An ellipse may be defined as the locus of a point, P, which moves such that the sum of its distances from two fixed points, F_1 and F_2, is constant.

F_1 and F_2 are the foci of the ellipse. PF_1 and PF_2 are called focal radii. P represents any point on the ellipse.

$$\left|\overline{PF_1} + \overline{PF_2}\right| = \text{a constant}$$

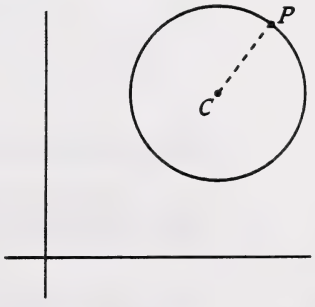
P_1 and P_2 are called the vertices of the ellipse.

$$\left|\overline{P_1P_2}\right| = \left|\overline{PF_1} + \overline{PF_2}\right|$$



A circle is the locus of a point, P, in a plane, which moves such that it is always the same distance from a fixed point called the centre, C. This distance is called the radius.

C is the centre of the circle.
CP is the radius.
P represents any point on the circle.



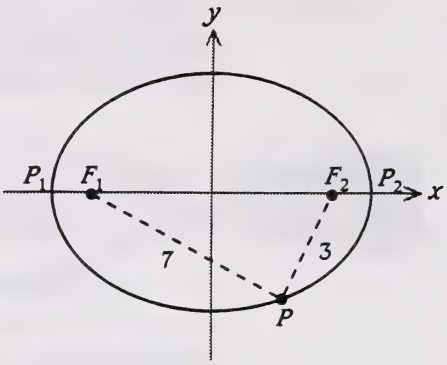
Example 1

Find the length of the line segment $\overline{P_1P_2}$ in the ellipse.

Solution

$$\left|\overline{PF_1} + \overline{PF_2}\right| = 7 + 3$$
$$= 10$$

The constant is 10. So, $\left|\overline{P_1P_2}\right| = 10$.



Example 2

A point $(6, y)$ is on the graph described by $x^2 + y^2 - 4x - 12y + 20 = 0$. Identify the type of conic this equation describes and determine the value of y. Verify your answer by using a graphing calculator or a computer graphing program.

Solution

$A = C = 1$ $\therefore$ the conic is a circle.

To determine the value of y, substitute $x = 6$ into the equation.

$$6^2 + y^2 - 4(6) - 12y + 20 = 0$$
$$36 + y^2 - 24 - 12y + 20 = 0$$
$$y^2 - 12y + 32 = 0$$
$$(y - 4)(y - 8) = 0$$
$$y - 4 = 0 \quad \text{or} \quad y - 8 = 0$$
$$y = 4 \qquad\qquad y = 8$$

There are two points on the circle where $x = 6$: $(6, 4)$ and $(6, 8)$.

Enter the equation into a graphing calculator or a computer graphing program to verify that it represents a circle that contains these two points.

TEXT

Additional Examples
Mathematics 12: pp. 95, 113, 114 (Example 1, Step 1 and Step 2)
Mathematics: Principles and Process 12: p. 321 (Example, down to fourth line of Step 3)
Holtmath 12: p. 165 (Example 1, Steps 1-4)

Exercises

1. Sketch an ellipse in which the foci are $F_1(-3,0)$ and $F_2(3,0)$ and the sum of the lengths of the focal radii is 10 units. Write an algebraic statement that could be used to derive the equation of the ellipse.

2. Write a definition for the ellipse in question 1 using the term *focal radii*.

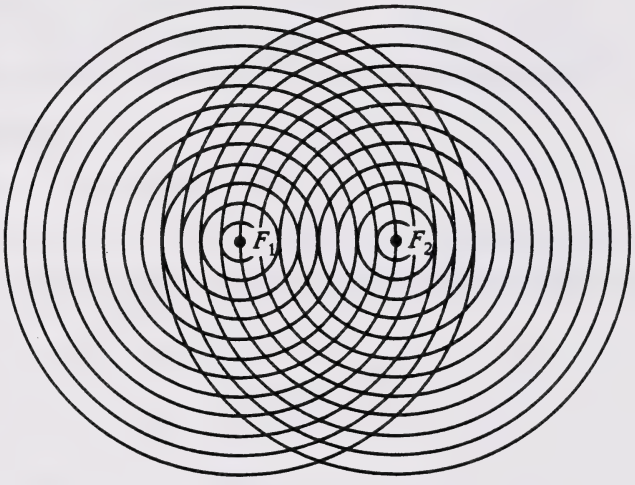
EXERCISE 7a SOLUTIONS are at the back of this workbook.

Double-Concentric Circle Grid

To sketch the locus of an ellipse, a special type of grid called the double-concentric grid can be used. Its use is explained in Video Segment 7b.

Exercises

3. On the double-concentric circle grid, plot points to indicate the locus of points that describes an ellipse constant sum is 11.



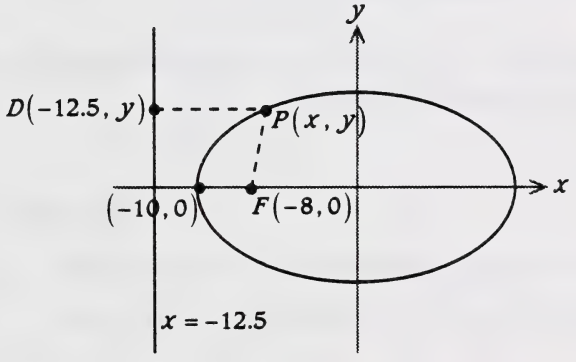
EXERCISE 7b SOLUTIONS are at the back of this workbook.

Relating the Eccentricity and Locus Definitions

There are actually **two** locus definitions for the ellipse: one involving a focus and directrix (the eccentricity definition) and the other involving two foci (the constant sum definition). This video refers to the latter as the locus definition and the former as the eccentricity definition.

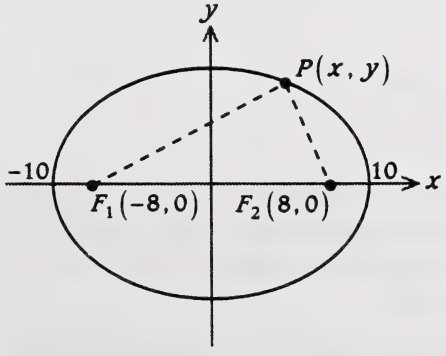
Three different sketches can be drawn to represent the ellipse discussed in Video Segment 7c. Any one of these can be used to derive the equation of the ellipse.

Eccentricity Definition with Focus on the Left



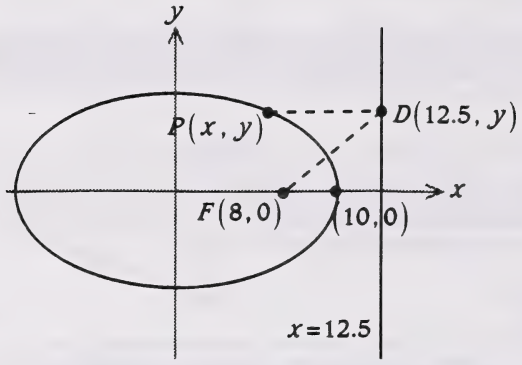
$$e = \frac{|\overline{PF}|}{|\overline{PD}|}$$
$$= \frac{4}{5}$$

Two Foci (Constant Sum) Definition



$$|\overline{PF_1} + \overline{PF_2}| = \text{a constant}$$
$$= 20$$

Eccentricity Definition with Focus on the Right



$$e = \frac{|\overline{PF}|}{|\overline{PD}|} = \frac{4}{5}$$

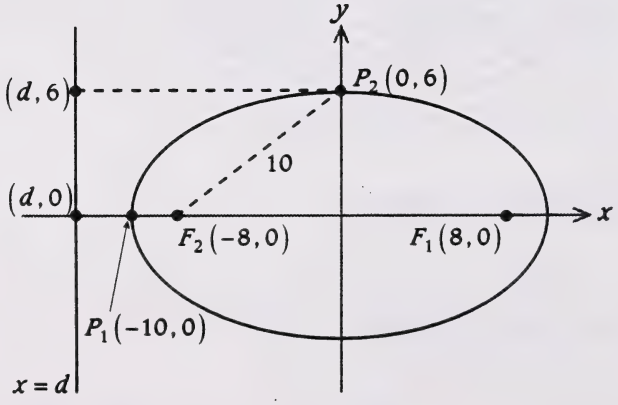
In the video segment, the student asks the instructor about the relation between the eccentricity definition and the constant sum definition. In the explanation, the foci, constant sum, eccentricity and equation of the directrix are all given and the relationship between the two definitions is verified.

Suppose only the foci and constant sum are given. Can the eccentricity and equation of a directrix be found? Yes! The following example shows how this is done.

Example

The foci of an ellipse are $F_1(8,0)$ and $F_2(-8,0)$ and the constant sum is 20. Find the eccentricity and the equation of the directrix.

Solution



The eccentricity, e, and equation of the directrix, $x = d$, are unknown. To determine the values of e and d, two equations are set up which involve the points P_1 and P_2 on the ellipse. P_1 is $(-10,0)$ because the constant sum is 20. P_2 is $(0,6)$ which can be found by applying the Pythagorean Theorem.

$$\frac{|\overline{P_1F}|}{|\overline{P_1D}|} = e \quad \text{and} \quad \frac{|\overline{P_2F}|}{|\overline{P_2D}|} = e$$

$$\frac{[-8-(-10)]}{(-10-d)} = e \qquad \frac{10}{(0-d)} = e$$

$$\frac{2}{(-10-d)} = e \qquad \frac{10}{-d} = e$$

$$\therefore \frac{2}{(-10-d)} = \frac{10}{-d}$$

$$-2d = -100 - 10d$$

$$8d = -100$$

$$d = -12.5$$

The equation of a directrix is $x = -12.5$.

$$\text{eccentricity}, e = \frac{|\overline{PF}|}{|\overline{PD}|}$$

$$= \frac{10}{|0-(-12.5)|} \quad \text{(using the point } P_2(0,6)\text{)}$$

$$= \frac{20}{25}$$

$$= \frac{4}{5}$$

Exercise

4. Given that the foci of an ellipse have coordinates $(\pm 3, 0)$ and the equation of the directrix is $x = \frac{25}{3}$, determine the constant sum locus definition and the eccentricity.

EXERCISE 7c SOLUTION is at the back of this workbook.

Deriving the Equation of a Circle

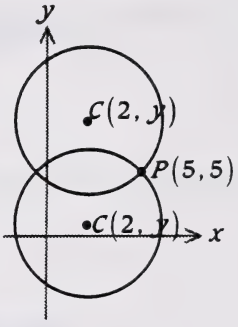
Example

The point $(5,5)$ is on a circle of radius five units. The x-coordinate of the centre is 2. Find the two possible locations for the centre, sketch and then state the domain and range of both circles.

Solution

CP = a constant

Using the distance formula:



$$\sqrt{(5-2)^2+(5-y)^2}=5$$
$$3^2+(5-y)^2=25$$
$$9+25-10y+y^2=25$$
$$y^2-10y+9=0$$
$$(y-1)(y-9)=0$$
$$y=1 \text{ or } y=9$$

The centre is located at $(2,1)$ or $(2,9)$.

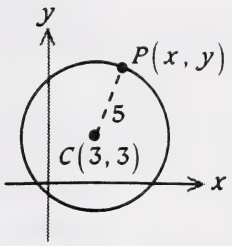
The domain of each circle: $\{-3 \le x \le 7\}$

Range of circle with centre $(2,9)$: $\{4 \le y \le 14\}$

Range of circle with centre $(2,1)$: $\{-4 \le y \le 6\}$

Exercise

5. State the domain and the range, and derive the equation of the circle to the right.



EXERCISE 7d SOLUTION is at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 98 (Questions 7, 8)
Mathematics: Principles and Process 12: p. 305 (Questions 5, 8a, 9), p. 307 (Question 4)
Holtmath 12: p. 156 (Questions 7b, 7c, 8)

Verifying the Locus Definition of an Ellipse

It is easy to verify that $9x^2 + 25y^2 - 225 = 0$ is the equation of an ellipse. Remember that for a quadratic relation to be an ellipse, it is necessary that $A \neq C$ and $AC > 0$. Both of these conditions are met in the equation $9x^2 + 25y^2 - 225 = 0$, so it is an ellipse.

In Video Segment 7e, $9x^2 + 25y^2 - 225 = 0$ is given as the equation of an ellipse with foci at $(-4, 0)$ and $(4, 0)$. Based on this information, verify the constant sum definition of this ellipse.

Watch Video Segment 7e.

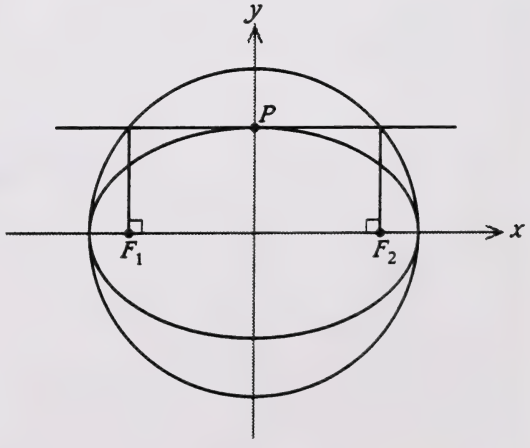
Exercises

Note: The following question is given to see if you can write a proof of the ruler and compass construction of locating the foci of an ellipse as shown in the video. The proof is **not** part of the Alberta curriculum requirements for Mathematics 30 and is for interest only. A proof is given in the Solutions at the back of the workbook.

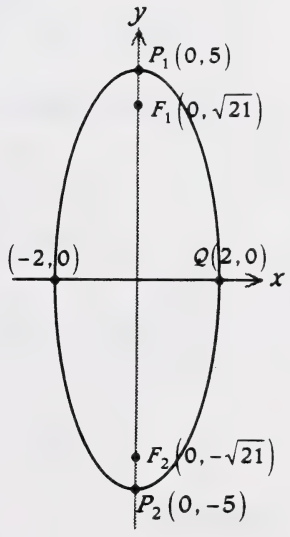
6. Prove the construction given by the instructor in the video segment on locating the two foci of an ellipse, that is, prove that the construction does locate the two foci.

Construction:

- Draw a circle that has the diameter as the major axis of the ellipse.
- Draw the tangent to the ellipse at P. This tangent cuts the circle at two points.
- Draw perpendiculars from these two points to the major axis. Points F_1 and F_2 are the two foci of the ellipse.



7. The ellipse shown is centred at the origin $\left|\overline{P_1P_2}\right| = 10$ units. The x-intercepts are $(\pm 2, 0)$. Verify the constant sum definition for this ellipse.



EXERCISE 7e SOLUTIONS are at the back of this workbook.

WHAT HAVE YOU LEARNED?

Return to the Learner Expectations at the beginning of the lesson and check off the points that you have mastered. Go back to the video and study those parts that you find difficult. Knowledge of the eccentricity and locus definitions of the ellipse is helpful when studying the hyperbola in **Lesson 8: Locus and Eccentricity of the Hyperbola.**

Solutions

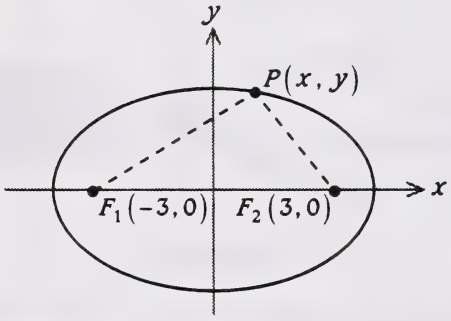
EXERCISE 7a

1. $\left|\overline{PF_1} + \overline{PF_2}\right|$ = a constant

$$\left|\sqrt{[x-(-3)]^2+(y-0)^2} + \sqrt{(x-3)^2+(y-0)^2}\right| = 10$$

or $\sqrt{(x+3)^2+y^2} + \sqrt{(x-3)^2+y^2} = 10$

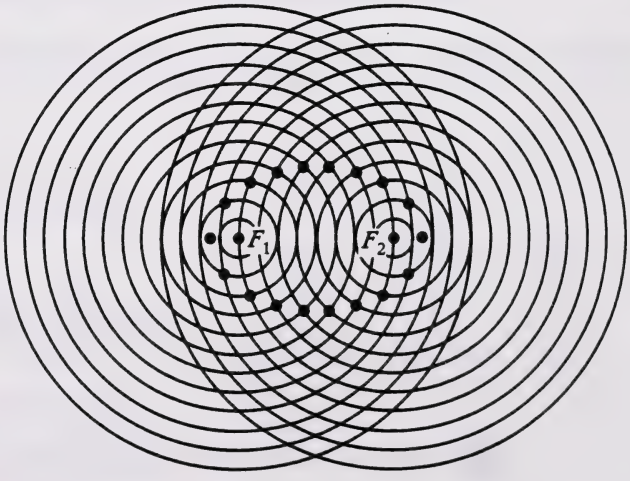
2. The sum of the lengths of the two focal radii in an ellipse is a constant amount.



EXERCISE 7b

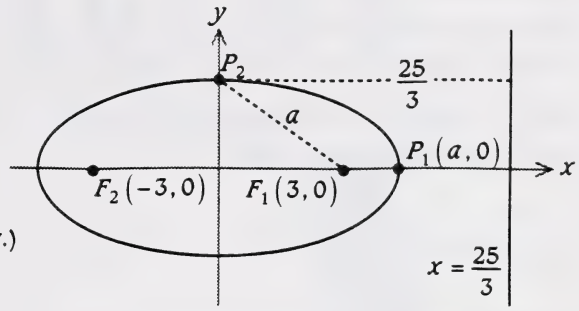
3. $\left|\overline{PF_1} + \overline{PF_2}\right| = 11$

$\overline{PF_1}$	$\overline{PF_2}$
1.5	9.5
2	9
3	8
4	7
5	6
6	5
7	4
8	3
9.5	1.5



EXERCISE 7c

4. $\dfrac{|\overline{P_1F_1}|}{|\overline{P_1D}|} = e$ and $\dfrac{|\overline{P_2F_1}|}{|\overline{P_2D}|} = e$



$$\therefore \frac{(a-3)}{\left(\frac{25}{3}-a\right)} = \frac{a}{\left(\frac{25}{3}\right)}$$

$$\left(\frac{25}{3}\right)(a-3) = a\left(\frac{25}{3}-a\right) \quad \text{(Cross multiply.)}$$

$$25(a-3) = 3a\left(\frac{25}{3}-a\right)$$

$$25a - 75 = 25a - 3a^2$$

$$3a^2 = 75$$

$$a^2 = 25$$

$$a = \pm 5$$

Constant sum locus definition:

$$|\overline{PF_1} + \overline{PF_2}| = 2a$$

$$= 10$$

Eccentricity, $e = \dfrac{|\overline{P_2F_1}|}{|\overline{P_2D}|}$

$$= \frac{5}{\left(\frac{25}{3}\right)}$$

$$= 5 \times \frac{3}{25}$$

$$= \frac{3}{5}$$

EXERCISE 7d

5. CP = a constant

Use the distance formula.

$$\sqrt{(x-3)^2 + (y-3)^2} = 5$$

$$(x-3)^2 + (y-3)^2 = 25 \quad \text{(Square both sides.)}$$

$$x^2 - 6x + 9 + y^2 - 6y + 9 = 25$$

$$x^2 + y^2 - 6x - 6y - 7 = 0$$

Domain: $\{-2 \le x \le 8\}$

Range: $\{-2 \le y \le 8\}$

EXERCISE 7e

6. Let the centre be $O(0,0)$.

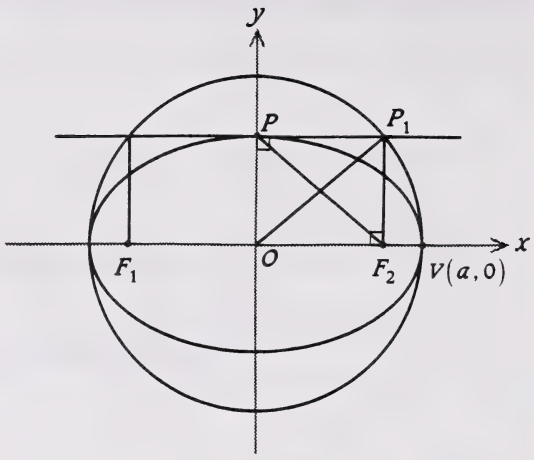
   Let the right vertex be $V(a,0)$.
   Therefore, the radius of the circle is a units.

   OPP_1F_2 is a rectangle. (Opposite angles are 90°.)
   Therefore, the diagonals are equal.

   $OP_1 = PF_2$
   But OP_1 is the radius of the circle.
   $\therefore OP_1 = a$.

   Hence, $PF_2 = a$.
   Therefore, F_2 must be a focus.

   Note: A similar proof exists for the other focus.



7. You must show $\left|\overline{PF_1} + \overline{PF_2}\right|$ = a constant for two points on the locus. Points P_1 and Q on the ellipse can be used as examples of any point P on the locus.

   $\left|\overline{P_1P_2}\right| = 10$ and the y - intercepts are ± 5.

$$\therefore \left|\overline{P_1F_1} + \overline{P_1F_2}\right| = \left|\left(5-\sqrt{21}\right)+\left[5-\left(-\sqrt{21}\right)\right]\right|$$
$$= \left|5-\sqrt{21}+5+\sqrt{21}\right|$$
$$= |10|$$
$$= 10$$

   $\left|\overline{QF_1}\right|$ and $\left|\overline{QF_2}\right|$ are focal radii of equal length.

$$\therefore \left|\overline{QF_1} + \overline{QF_2}\right| = 5+5$$
$$= 10$$

   Thus, $\left|\overline{PF_1} + \overline{PF_2}\right|$ = a constant is verified.

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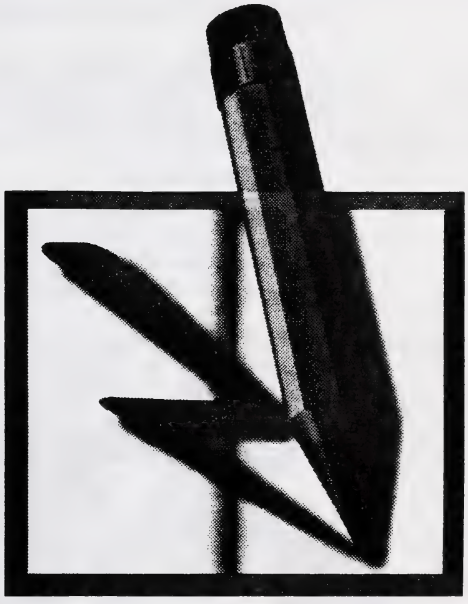
MATH FACTOR

QUADRATIC RELATIONS



Lesson 8: Locus and Eccentricity of the Hyperbola

Student Workbook

Produced by

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for

Alberta EDUCATION

This guide accompanies **Locus and Eccentricity of the Hyperbola** from the Math Factor series.

Writer: Gordon Mallett
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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Quadratic Relations: Locus and Eccentricity of the Hyperbola

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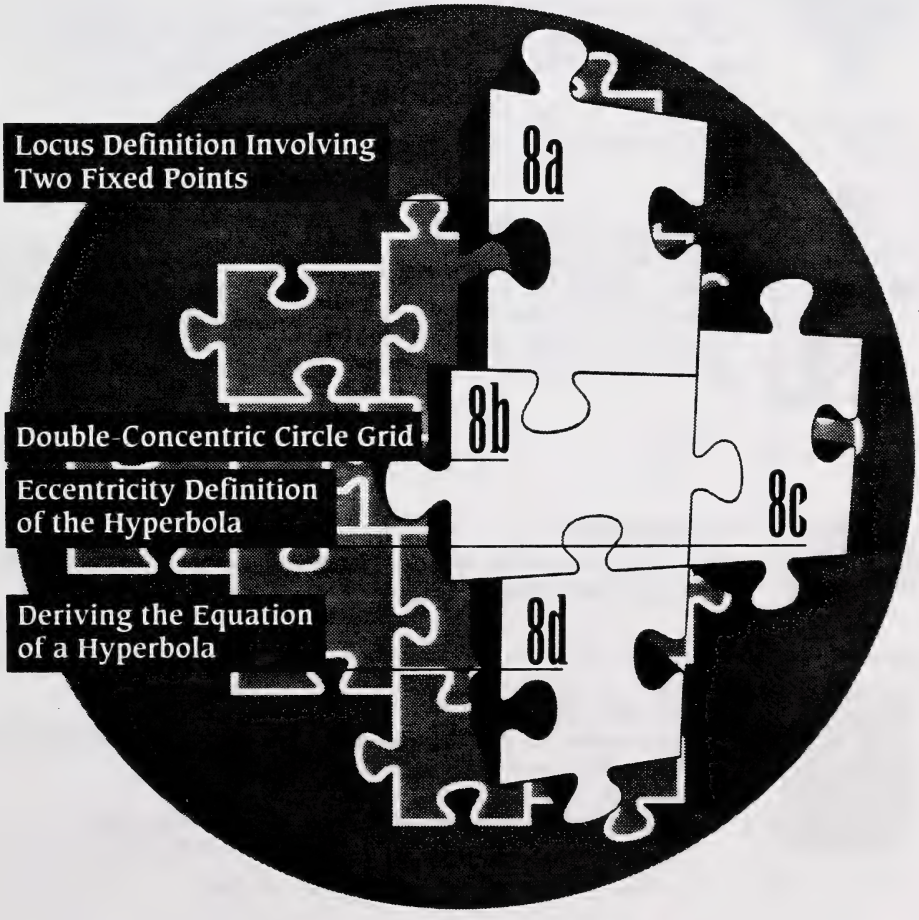
Lesson 8: LOCUS AND ECCENTRICITY OF THE HYPERBOLA

LEARNER EXPECTATIONS

By the end of Lesson 8 you should be able to

- recognize that there are several approaches that can be used to study the shape of a hyperbola
- analyze hyperbolas defined in terms of the two fixed points
- analyze hyperbolas defined in terms of eccentricity
- use a double-concentric circle grid and a calculator or computer graphing program to draw and analyze the graphs of hyperbolas

This lesson has four parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

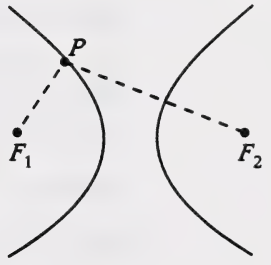
Locus Definition Involving Two Fixed Points

Of all the conic sections, only the hyperbola and ellipse can be defined in terms of a locus involving two fixed points.

Watch the Introduction and Video Segment 8a.

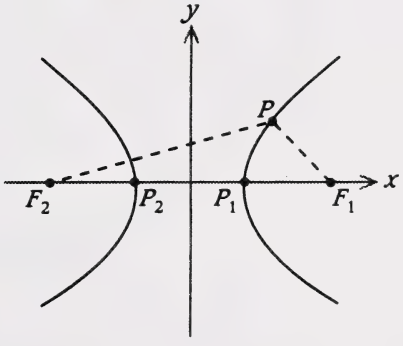
A hyperbola may be defined as the locus of a point, P, which moves such that the absolute value of the difference of its distances from two fixed points, F_1 and F_2, is a constant.

F_1 and F_2 are the foci of the hyperbola.
PF_1 and PF_2 are called focal radii.
P represents any point on the hyperbola.



$\left|\overline{PF_1} - \overline{PF_2}\right|$ = a constant

$$\left|\overline{P_1P_2}\right| = \left|\overline{PF_1} - \overline{PF_2}\right|$$

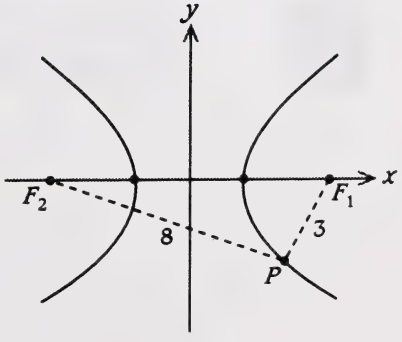


Example 1

Sketch a hyperbola in which $\left|\overline{PF_1} - \overline{PF_2}\right|$ is 5.

Solution

$$\begin{aligned}\left|\overline{PF_1} - \overline{PF_2}\right| &= |3 - 8| \\ &= |-5| \\ &= 5\end{aligned}$$



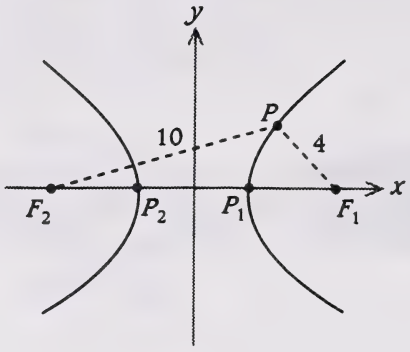
Example 2

Determine $\left|\overline{P_1P_2}\right|$ in the hyperbola shown.

Solution

$$\begin{aligned}\left|\overline{PF_1} - \overline{PF_2}\right| &= |4 - 10| \\ &= |-6| \\ &= 6\end{aligned}$$

The constant is 6, therefore, $\left|\overline{P_1P_2}\right| = 6$.



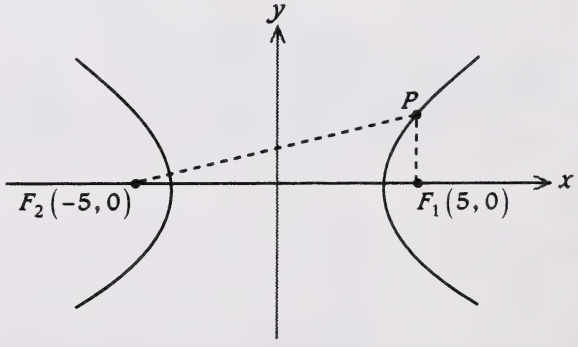
Example 3

Draw a sketch of the hyperbola in which the foci are $F_1(5,0)$ and $F_2(-5,0)$ and the difference of the focal radii is eight units. Write an algebraic statement that could be used to derive the equation of the hyperbola.

Solution

$\left|\overline{PF_1} - \overline{PF_2}\right|$ = a constant

$$\left|\sqrt{(x-5)^2 + y^2} - \sqrt{(x+5)^2 + y^2}\right| = 8 \text{ or } \sqrt{(x-5)^2 + y^2} - \sqrt{(x+5)^2 + y^2} = 8$$



TEXT

Additional Examples
Mathematics 12: pp. 119-120
Mathematics: Principles and Process 12: p. 329 (down to fourth line of Step 3)
Holtmath 12: p. 172

Exercises

1. Write a definition for a hyperbola using the term *focal radii*.

2. Sketch the hyperbola in which the foci are $F_1(0, 3)$ and $F_2(0, -3)$ and the difference of the focal radii is four units. Write an algebraic statement that could be used to derive the equation of this hyperbola.

EXERCISE 8a SOLUTIONS are at the back of this workbook.

TEXT

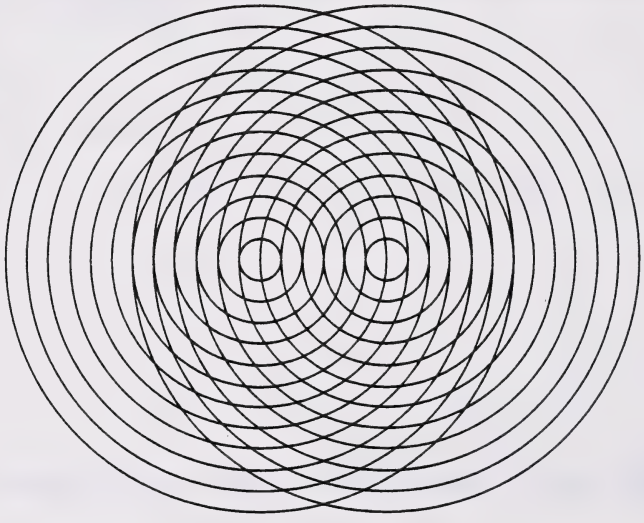
Additional Practice
Mathematics 12: p. 126 (Question 5), p. 130 (Question 5)

Double-Concentric Circle Grid

To sketch the locus of a hyperbola, a special type of grid called the double-concentric grid can be used. Its use is explained in Video Segment 8b.

Exercises

3. a. On the double-concentric circle grid given, plot points to indicate the locus of points in which the constant difference from two fixed points is three. Join the points with a smooth curve.



b. How far apart are the two vertices? How is this value related to the constant difference?

c. On the same grid, plot the locus of points where the constant difference from two fixed points is four.

d. Describe the differences between the graphs of the two hyperbolas.

EXERCISE 8b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics: Principles and Process 12: p. 320 (Question 2)

Eccentricity Definition of the Hyperbola

The ratio of $|\overline{PF}|$ to $|\overline{PD}|$ is referred to as the eccentricity, e, of a conic section. For the hyperbola, the eccentricity is always greater than 1.

$$\text{Eccentricity, } e = \frac{|\overline{PF}|}{|\overline{PD}|}$$

Conic	Eccentricity
ellipse	$0 < e < 1$
parabola	$e = 1$
hyperbola	$e > 1$

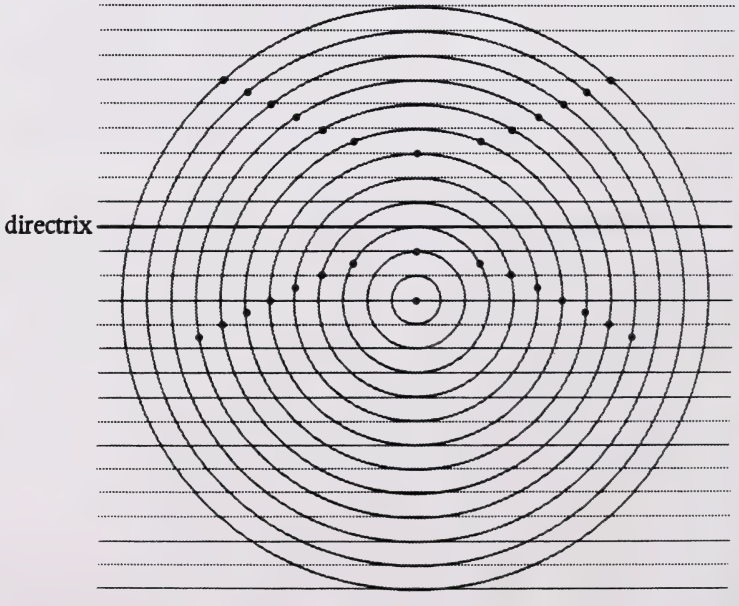
If a hyperbola is defined in terms of eccentricity, circle-line paper can be used to sketch its locus of points. Video Segment 8c explains how this is done.

Example

In a hyperbola where the eccentricity is 2, the distance from the focus to the directrix is three units. On a circle-line grid, draw the directrix and several points on the locus.

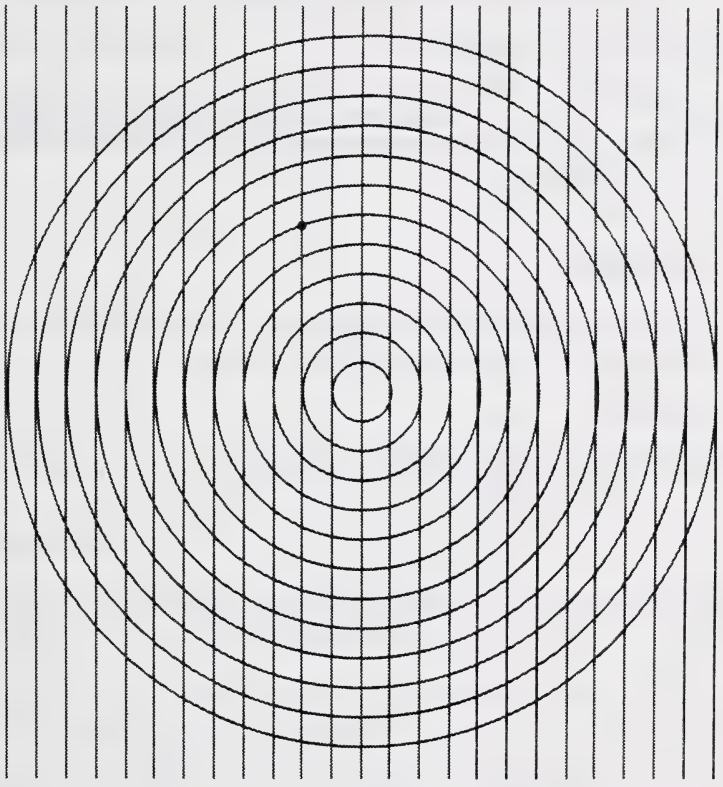
Solution

$\lvert\overline{PF}\rvert$	$\lvert\overline{PD}\rvert$
2	1
3	1.5
4	2
5	2.5
6	3
7	3.5
8	4
9	4.5



Exercises

4. The point shown belongs to a hyperbola with an eccentricity of 3. Locate the hyperbola's directrix, focus and two vertices.



5. On the grid in question 4, locate several points that are more than six units from the focus and that lie on the left branch of the hyperbola.

EXERCISE 8c SOLUTIONS are at the back of this workbook.

Deriving the Equation of a Hyperbola

Challenge yourself

Derive the equation of a hyperbola with focus $F(4, 2)$, directrix $x = 1$ and eccentricity 2.

MATH FACTOR

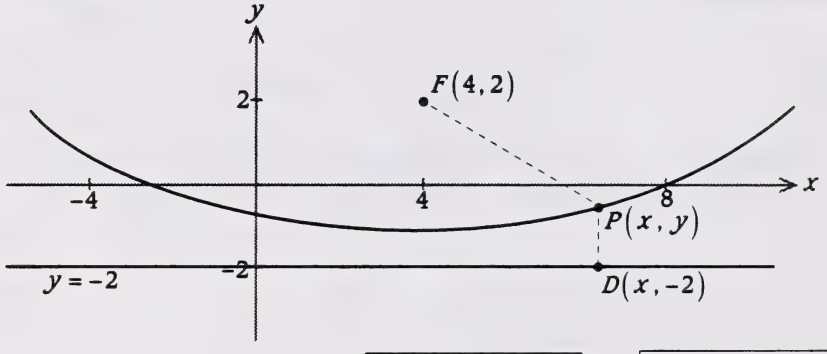
Watch Video Segment 8d.

Example

Determine the equation of the locus where the eccentricity is 3, directrix is $y = -2$ and focus is $F(4, 2)$. Verify the equation of the locus.

Solution

Only one of the two branches of the hyperbola is shown in this sketch.



$$e = 3$$

$$\therefore \frac{|\overline{PF}|}{|\overline{PD}|} = 3$$

$$|\overline{PF}| = 3|\overline{PD}|$$

$$\sqrt{(x-4)^2 + (y-2)^2} = 3\sqrt{(x-x)^2 + [y-(-2)]^2}$$

$$\left[(x-4)^2 + (y-2)^2\right] = 9(y+2)^2$$

$$x^2 - 8x + 16 + y^2 - 4y + 4 = 9\left(y^2 + 4y + 4\right)$$

$$x^2 - 8y^2 - 8x - 40y - 16 = 0$$

Algebraically, the equation can be verified by using any point, P, on the locus and showing that $\frac{|\overline{PF}|}{|\overline{PD}|} = 3$.

Find the x-intercepts.

$$x^2 - 8(0)^2 - 8x - 40(0) - 16 = 0$$
$$x^2 - 8x - 16 = 0$$
$$x = \frac{\left[8 \pm \sqrt{64 + 64}\right]}{2}$$
$$= \frac{\left(8 \pm 8\sqrt{2}\right)}{2}$$
$$= 4 \pm 4\sqrt{2}$$

Now take $\left(4 + 4\sqrt{2}, 0\right)$ as the point P.

$$|\overline{PF}| = \sqrt{\left[4 - \left(4 + 4\sqrt{2}\right)\right]^2 + 2^2}$$
$$= \sqrt{\left(-4\sqrt{2}\right)^2 + 2^2}$$
$$= \sqrt{32 + 4}$$
$$= 6$$

$$|\overline{PD}| = 2$$

Thus, $\frac{|\overline{PF}|}{|\overline{PD}|} = \frac{6}{2} = 3$, which verifies the locus.

Exercises

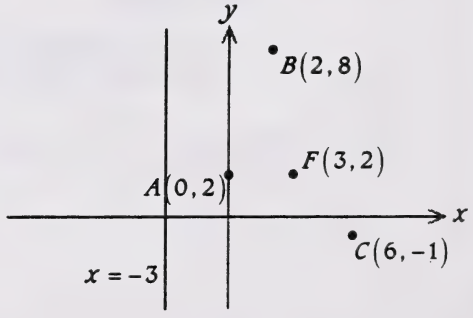
6. Sketch and write the equation of the locus of points that are three times as far from the point $P(5, 2)$ as they are from the line $x = -3$.

7. Determine the coordinates of the two vertices of the hyperbola in question 6.

8. Determine the domain and range of the hyperbola in question 6.

9. If the hyperbola in question 6 were defined in terms of two foci, determine the value of the constant difference.

10. In the diagram given, points $A(0,2)$, $B(2,8)$ and $C(6,-1)$ each lie on either a hyperbola, a parabola or an ellipse. If the focus and the directrix for each of these quadratic relations is $(3,2)$ and $x=-3$, then determine which quadratic relation each of the points lies on.



11. A hyperbola has a directrix of $y = 2$, one focus at $(0, 10)$ and an eccentricity of $\frac{5}{3}$. Determine the second focus and another directrix that could be used to describe this hyperbola.

EXERCISE 8d SOLUTIONS are at the back of this workbook.

WHAT HAVE YOU LEARNED?

Return to the Learner Expectations at the beginning of the lesson and check off the points that you have mastered. Go back to the video and study those parts that you find difficult. Knowledge of locus definitions and eccentricity is important for continuing on to **Lesson 9: Identification, Connections and Applications of Quadratic Relations**.

Solutions

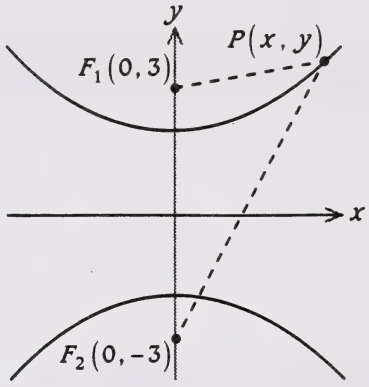
EXERCISE 8a

1. In a hyperbola, the difference in the lengths of the two focal radii is a constant amount.

2. $\left|\overline{PF_1} - \overline{PF_2}\right| = \text{constant}$

$$\left|\sqrt{\left(x^2 + (y-3)^2\right)} - \sqrt{x^2 + (y+3)^2}\right| = 4$$

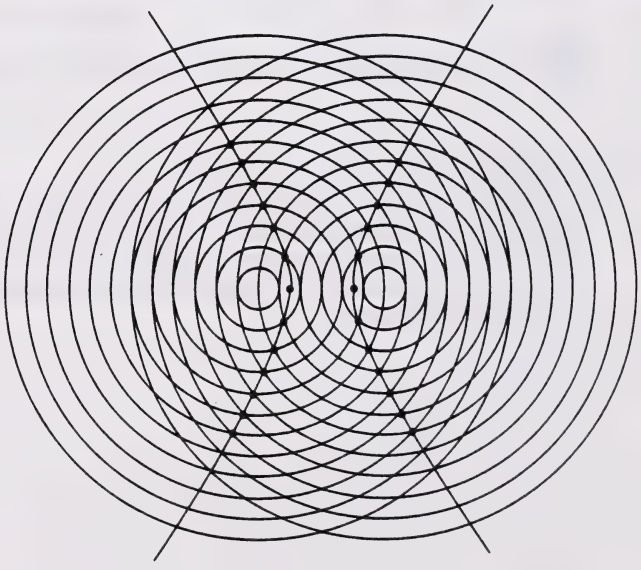
or $$\sqrt{\left(x^2 + (y-3)^2\right)} - \sqrt{x^2 + (y+3)^2} = 4$$



EXERCISE 8b

3. a.

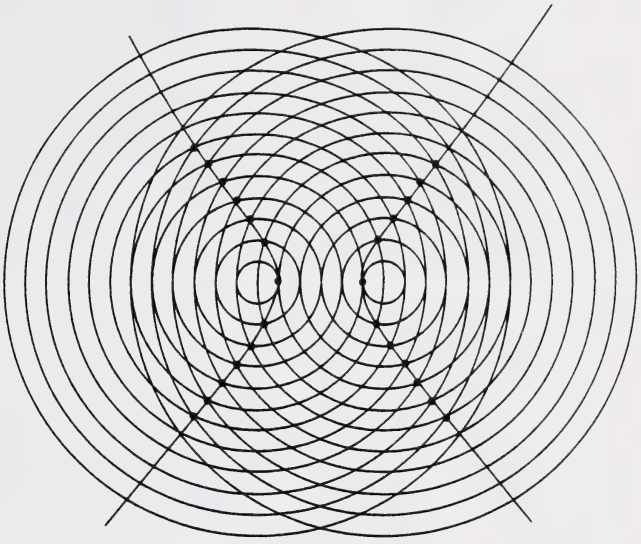
$\overline{PF_2}$	$\overline{PF_1}$
4.5	1.5
5	2
6	3
7	4
8	5
9	6
10	7



b. The vertices are $4.5 - 1.5 = 3$ units apart and this is equal to the constant difference.

c.

$\lvert\overline{PF_2}\rvert$	$\lvert\overline{PF_1}\rvert$
5	1
6	2
7	3
8	4
9	5
10	6
11	7



d. The hyperbola with constant difference of 4 has its vertices further apart than the hyperbola with constant difference of 3. The hyperbola with constant difference of 4 is narrower and curves more around the focus.

EXERCISE 8c

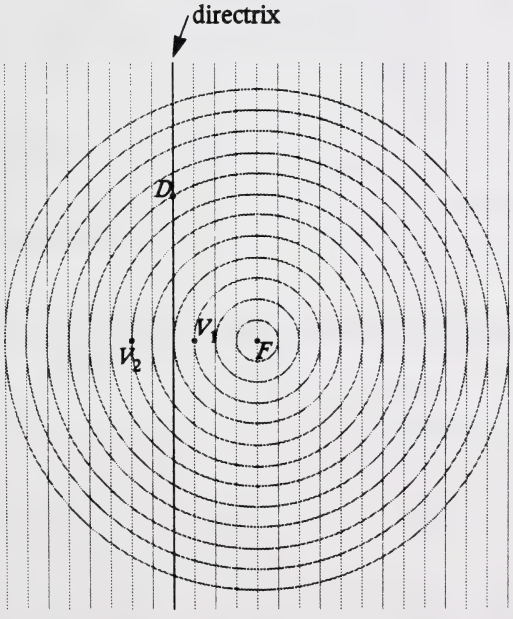
4. $\dfrac{\lvert\overline{PF}\rvert}{\lvert\overline{PD}\rvert} = 3$

$\therefore \dfrac{6}{\lvert\overline{PD}\rvert} = 3$

$6 = 3\lvert\overline{PD}\rvert$

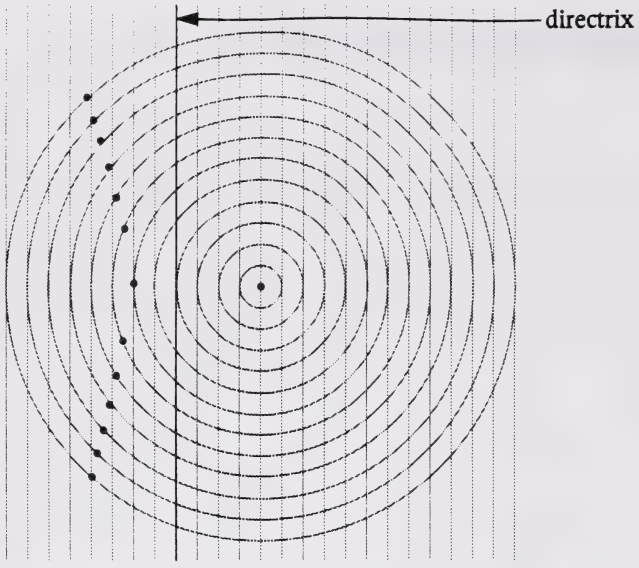
$\lvert\overline{PD}\rvert = 2$

	$\lvert\overline{PF}\rvert$	$\lvert\overline{PD}\rvert$
V_1	3	1
V_2	6	2



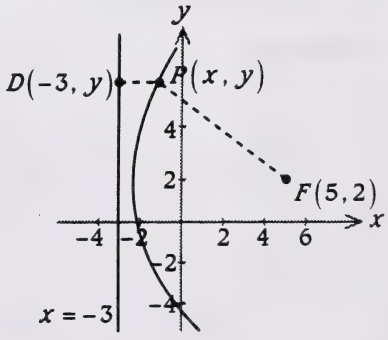
5.

$\overline{PF}$	$\overline{PD}$
7	$\frac{7}{3}$
8	$\frac{8}{3}$
9	3
10	$\frac{10}{3}$
11	$\frac{11}{3}$
12	4



EXERCISE 8d

6.
$$\left|\overline{PF}\right| = 3\left|\overline{PD}\right|$$
$$\sqrt{(x-5)^2+(y-2)^2} = 3\sqrt{[x-(-3)]^2+(y-y)^2}$$
$$(x-5)^2+(y-2)^2 = 9(x+3)^2$$
$$x^2-10x+25+y^2-4y+4 = 9(x^2+6x+9)$$
$$x^2-10x+25+y^2-4y+4 = 9x^2+54x+81$$
$$-8x^2+y^2-64x-4y-52 = 0$$
$$\therefore 8x^2-y^2+64x+4y+52 = 0$$



7. The y-coordinate of each vertex is 2. Substitute $y = 2$ in $8x^2 - y^2 + 64x + 4y + 52 = 0$.

$$8x^2-(2)^2+64x+4(2)+52 = 0$$
$$8x^2+64x+56 = 0$$
$$x^2+8x+7 = 0$$
$$(x+7)(x+1) = 0$$
$$x = -1 \text{ or } x = -7$$

The vertices are $(-1, 2)$ and $(-7, 2)$.

8. Domain: $\{x \geq -1 \text{ or } x \leq -7, x \in R\}$
Range: $y \in R$

9. $\left|\overline{P_1P_2}\right| = \left|-7-(-1)\right|$

$= \left|-6\right|$

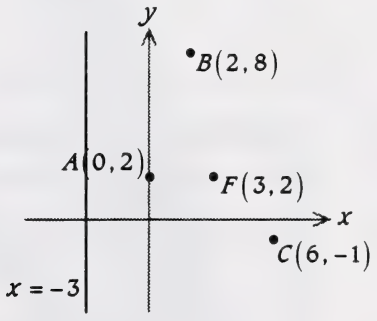
$= 6$ units

$\therefore$ constant $= 6$

10. $\left|\overline{AF}\right| = 3$ and $\left|\overline{AD}\right| = 3$

$\dfrac{\left|\overline{AF}\right|}{\left|\overline{AD}\right|} = 1,\ e = 1$

Therefore, A lies on a parabola.



$\left|\overline{BF}\right| = \sqrt{1^2+6^2}$ and $\left|\overline{BD}\right| = 5$

$= \sqrt{37}$

$\dfrac{\left|\overline{BF}\right|}{\left|\overline{BD}\right|} = \dfrac{\sqrt{37}}{5},\ e \doteq 1.2$

Therefore, B lies on a parabola.

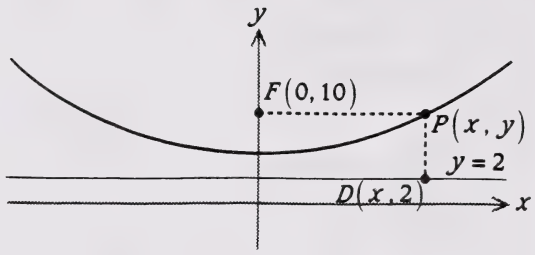
$\left|\overline{CF}\right| = \sqrt{3^2+3^2}$ and $\left|\overline{CD}\right| = 9$

$= \sqrt{18}$

$\dfrac{\left|\overline{CF}\right|}{\left|\overline{CD}\right|} = \dfrac{\sqrt{18}}{9},\ e \doteq 0.47$

Therefore, C lies on an ellipse.

11. Let P be any point (x, y) on the locus.



$$\frac{|\overline{PF}|}{|\overline{PD}|} = \frac{5}{3}$$

$$\frac{\sqrt{x^2 + (y-10)^2}}{\sqrt{(x-x)^2 + (y-2)^2}} = \frac{5}{3}$$

$$9\left(x^2 + y^2 - 20y + 100\right) = 25\left(y^2 - 4y + 4\right)$$

$$9x^2 + 9y^2 - 180y + 900 = 25y^2 - 100y + 100$$

$$9x^2 - 16y^2 - 80y + 800 = 0$$

The vertices are the y-intercepts. (Let $x = 0$.)

$$-16y^2 - 80y + 800 = 0$$

$$y^2 + 5y - 50 = 0$$

$$(y-5)(y+10) = 0$$

Therefore, the vertices are $(0, 5)$ and $(0, -10)$.

The distance from vertex to focus is 5. Therefore, the other focus is $(0, -15)$.

The distance from vertex to directrix is 3. Therefore, the equation of the other directrix is $y = -7$.

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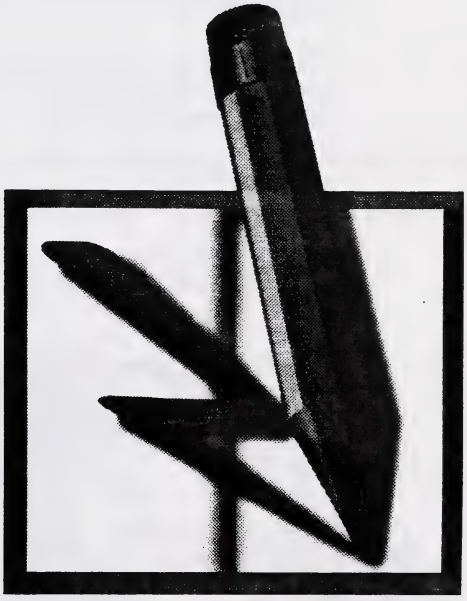
MATH FACTOR

QUADRATIC RELATIONS



Lesson 9:
Identification, Connections and Applications of Quadratic Relations

Student Workbook

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This guide accompanies **Identification, Connections and Applications of Quadratic Relations** from the Math Factor series.

Writer: Jim Ashton
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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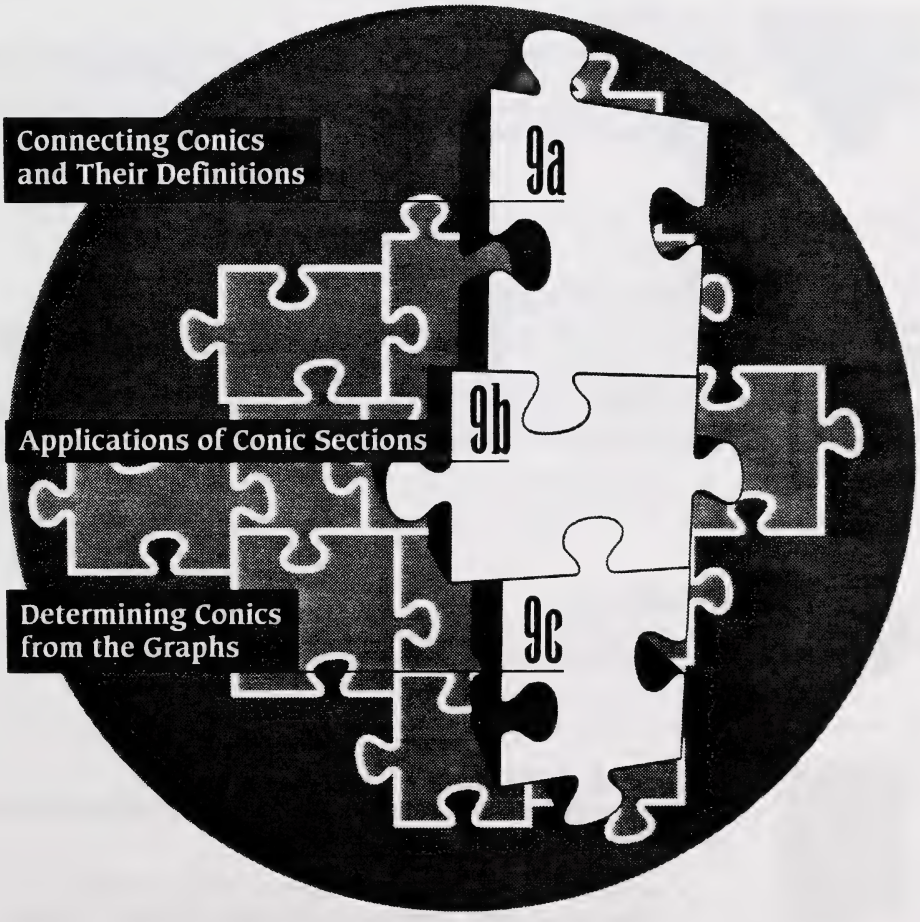
Lesson 9: IDENTIFICATION, CONNECTIONS AND APPLICATIONS OF QUADRATIC RELATIONS

LEARNER EXPECTATIONS

By the end of Lesson 9 you should be able to

- identify examples of the four types of conic, given either an equation or a description
- make connections between the types of quadratic relations and their definitions
- solve a variety of problems that can be represented by quadratic relations

This lesson has three parts:



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Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

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Student Workbook:

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Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

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- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Connecting Conics and Their Definitions

Watch the Introduction and Video Segment 9a.

Example 1

Match each diagram with one of the following possible equations.

- $x^2 + y^2 - 3x + 4y - 56 = 0$
- $y^2 - 3x - 4y = 21$
- $4x^2 + 3y^2 = 45$

Diagram A

Diagram B

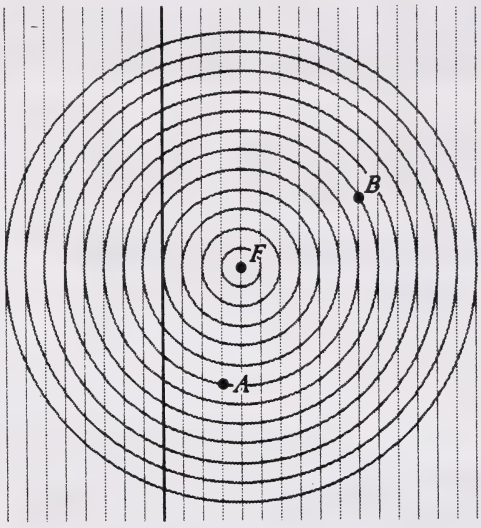
Diagram C

Solution

Diagram A represents $4x^2 + 3y^2 = 45$ since both represent an ellipse.
Diagram B represents $x^2 + y^2 - 3x + 4y - 56 = 0$ since both represent a circle.
Diagram C represents $y^2 - 3x - 4y = 21$ since both represent a parabola.

Example 2

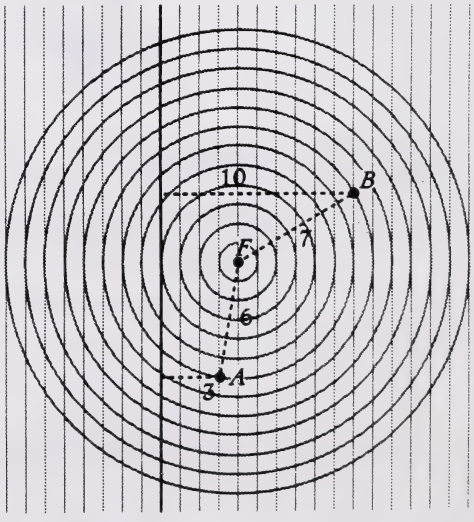
If F is the focus and the dark line is the directrix, then on which conic does point A lie? On which conic does point B lie?



Solution

For A, the eccentricity is $\frac{6}{3} = 2$;
thus, A lies on a hyperbola.

For B, the eccentricity is $\frac{7}{10} = 0.7$;
thus, B lies on an ellipse.



Exercises

1. Write an equation that could represent the conic formed when the cutting plane of a cone is parallel to the cone's axis of symmetry.

2. Write an equation that could represent the conic formed when the cutting plane of a cone is parallel to the cone's generator.

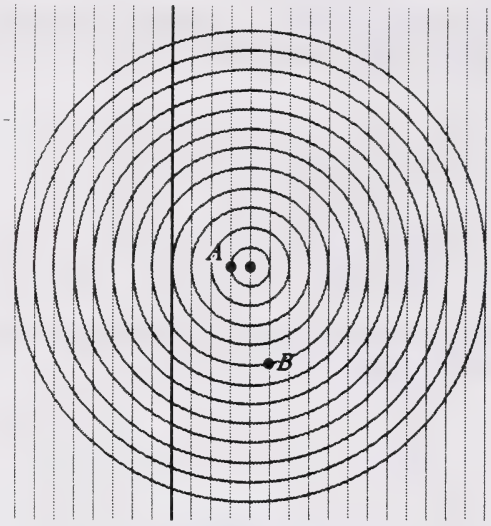
3. The equation $x^2 + 3y^2 + 4x - 3y - 56 = 0$ represents a conic section formed when the cutting plane of a cone is

   A. perpendicular to the axis of symmetry
   B. between the generator and the line that is perpendicular to the axis of symmetry
   C. parallel to the generator
   D. between the generator and the axis of symmetry of the cone

4. The equation $x^2 - 4y^2 - 49 = 0$ represents a conic section formed when the cutting plane of a cone is

   A. perpendicular to the axis of symmetry
   B. between the generator and the line that is perpendicular to the axis of symmetry
   C. parallel to the generator
   D. between the generator and the axis of symmetry

5. Use the diagram to answer the questions that follow.



a. *A* lies on what type of conic?

b. *B* lies on what type of conic?

EXERCISE 9a SOLUTIONS are at the back of this workbook.

Applications of Conic Sections

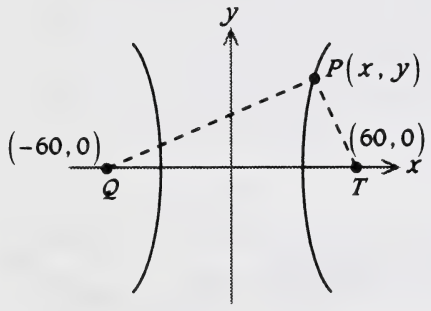
MATH FACTOR

Watch Video Segment 9b.

EXCELLENCE

Example

The loran (**long range**) system of navigation is used for determining the locations of ships and planes. Radio signals are sent by two stations, Q and T, as shown in the diagram. Point P represents the location of a ship on one branch of the hyperbola, while Q and T serve as focal points for the hyperbolic path. If the sending stations are 120 km apart and the ship is 40 km closer to T than it is to Q, determine the equation of the hyperbolic path on which the ship is located.



Solution

$$\left|\overline{PQ} - \overline{PT}\right| = 40$$

$$\sqrt{\left[x-(-60)^2\right]+(y-0)^2} - \sqrt{(x-60)^2+(y-0)^2} = 40$$

$$\sqrt{(x+60)^2+y^2} = 40 + \sqrt{(x-60)^2+y^2}$$

Square both sides.

$$x^2+120x+3600+y^2 = 1600+80\sqrt{(x-60)^2+y^2}+x^2-120x+3600+y^2$$

$$240x-1600 = 80\sqrt{(x-60)^2+y^2}$$

$$3x-20 = \sqrt{(x-60)^2+y^2}$$

Square both sides.

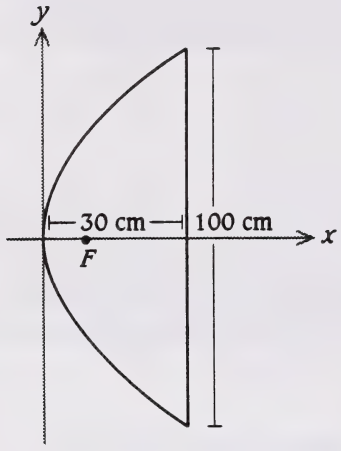
$$9x^2-120x+400 = x^2-120x+3600+y^2$$

$$8x^2-y^2-3200 = 0$$

The equation that describes the hyperbolic path is $8x^2 - y^2 - 3200 = 0$.

Exercises

6. A searchlight reflector is designed so that a cross section through its axis is a parabola and the light source is at the focus. The width of the reflector is 100 cm and the reflector is 30 cm deep. Find the distance from the vertex to the light source.

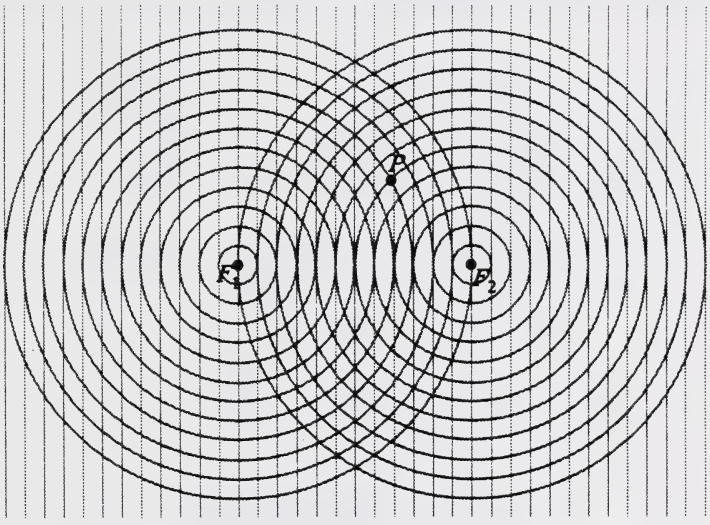


EXERCISE 9b SOLUTIONS are at the back of this workbook.

Determining Conics from the Graphs

Challenge yourself

Determine what type of conic will be formed by the points in the following graph.

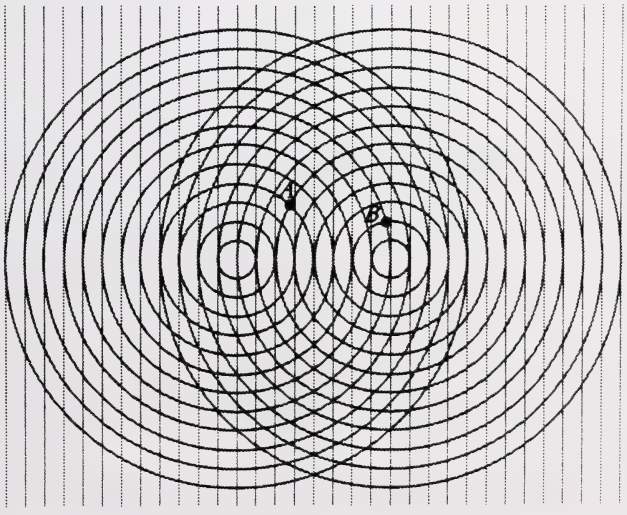


MATH FACTOR

Watch Video Segment 9c.

Example

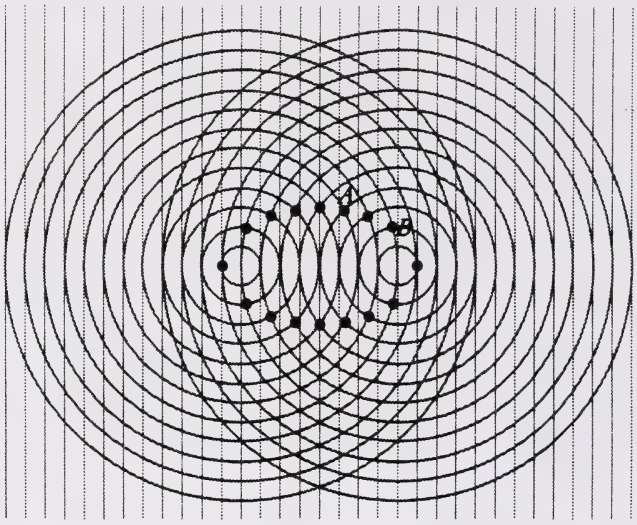
For the graph shown, determine which type of conic can be drawn with the given points and then complete the graph.



Solution

Point A is six units from one focus and four units from the other. Point B is eight units from one focus and two units from the other. Thus, the constant sum is 10 and this is an ellipse.

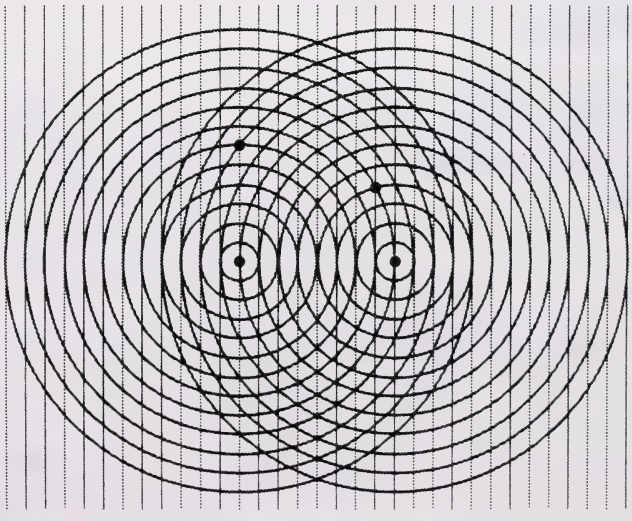
The completed graph is as follows:



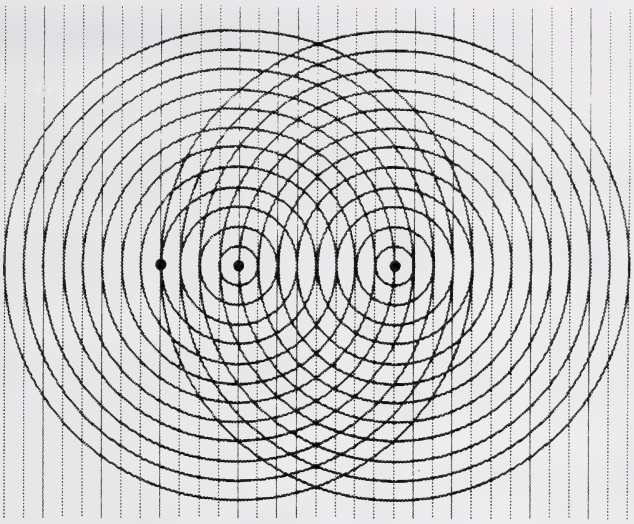
Exercises

7. Identify the type of conic that can be drawn with the given points on the graphs and then complete the graphs. The centres of the smallest circles are the two foci.

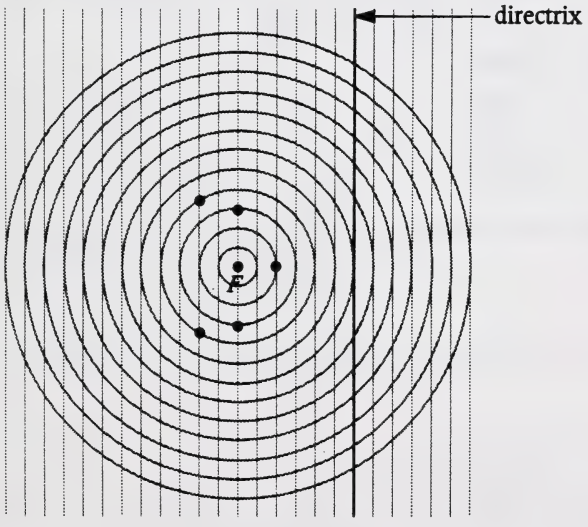
   a.



b.



8. Look at the graph and answer the questions that follow.



a. If F is the focus and the thick line is the directrix, then determine what type of conic these points will form and complete the graph.

b. Determine an equation to represent this conic.

EXERCISE 9c SOLUTIONS are at the back of this workbook.

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. In the next lesson, you will review *Quadratic Relations*.

Solutions

EXERCISE 9a

1. Answers will vary. For example, $x^2 - y^2 - 36 = 0$. (Any equation of a hyperbola where either A or C is negative.)

2. Answers will vary. For example: $x^2 - 4x + 3y + 5 = 0$. (Any equation of a parabola where A or C is 0.)

3. B 4. D

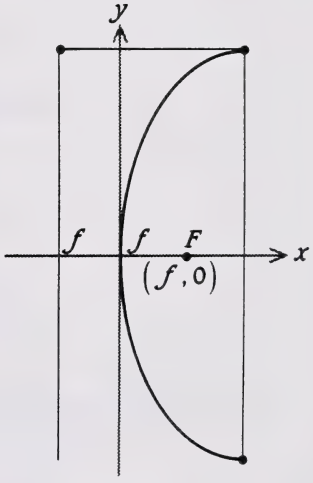
5. a. Since $e = \frac{1}{3}$, A lies on an ellipse. b. Since $e = \frac{5}{5} = 1$, B lies on a parabola.

EXERCISE 9b

6. Let $(f, 0)$ be the coordinates of the focus, F.

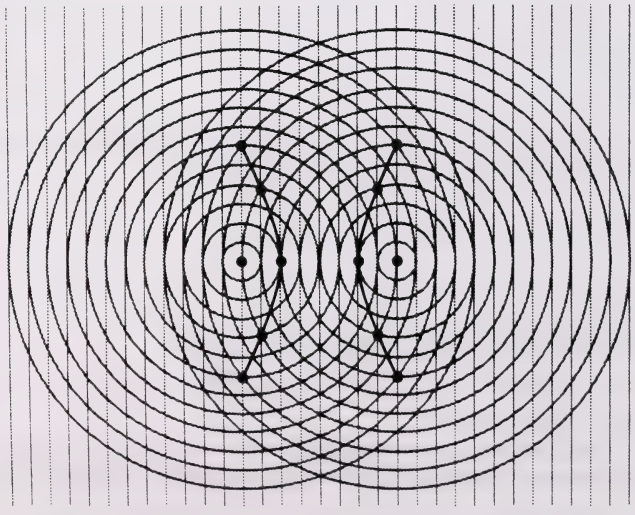
$$\sqrt{[30-(-f)]^2 + (50-50)^2} = \sqrt{(30-f)^2 + (50-0)^2}$$
$$(30+f)^2 = (30-f)^2 + 50^2$$
$$900 + 60f + f^2 = 900 - 60f + f^2 + 2500$$
$$120f = 2500$$
$$f = \frac{2500}{120}$$
$$\doteq 20.8 \text{ cm}$$

The focus is approximately 20.8 cm from the vertex.

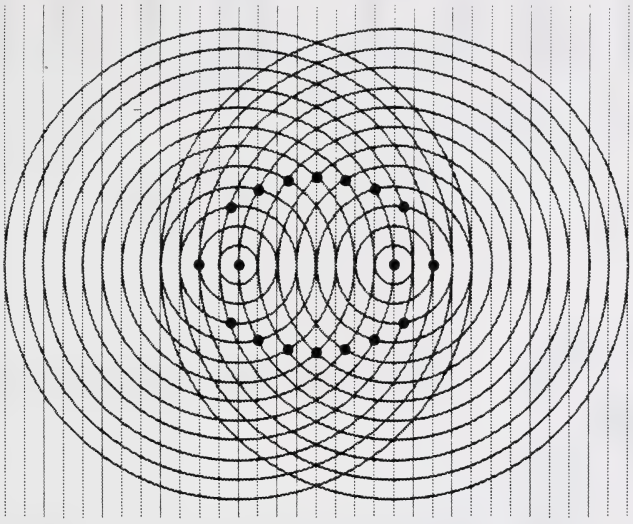


EXERCISE 9c

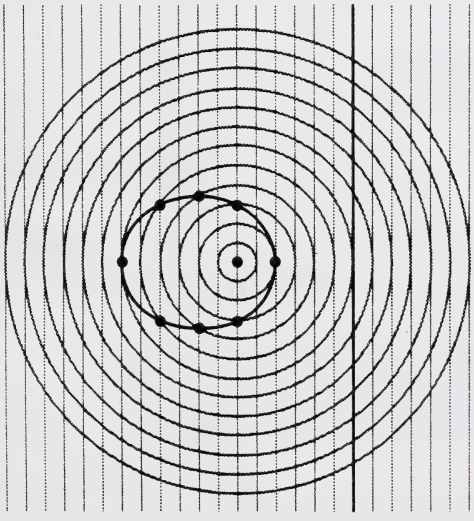
7. a. The graph is a hyperbola with a constant difference of 4.



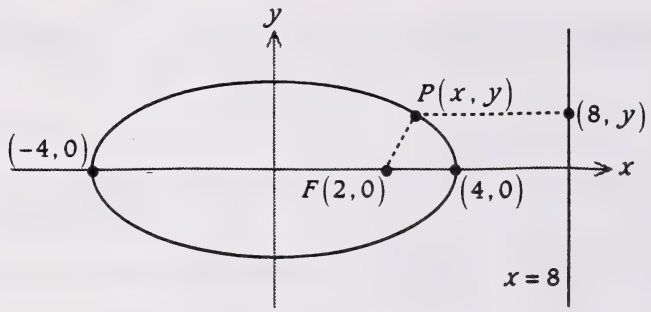
b. The graph is an ellipse with a constant sum of 12.



8. a. Since $e = \frac{1}{2}$, this is an ellipse.



b.



$$\frac{|\overline{PF}|}{|\overline{PD}|} = \frac{1}{2}$$

$$2|\overline{PF}| = |\overline{PD}|$$

$$2\sqrt{\left[x-(2)^2\right]+(y-0)^2} = \sqrt{(x-8)^2+(y-y)^2}$$

$$4\left(x^2-4x+4+y^2\right) = x^2-16x+64+0$$

$$4x^2-16x+16+4y^2 = x^2-16x+64$$

$$3x^2+4y^2-48 = 0$$

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MATH FACTOR

QUADRATIC RELATIONS

Lesson 10:
Review

Student Workbook

Produced by

THE EDUCATION STATION

for

Alberta
EDUCATION

This guide accompanies **Review** from the Math Factor series.

Writer: Jim Ashton
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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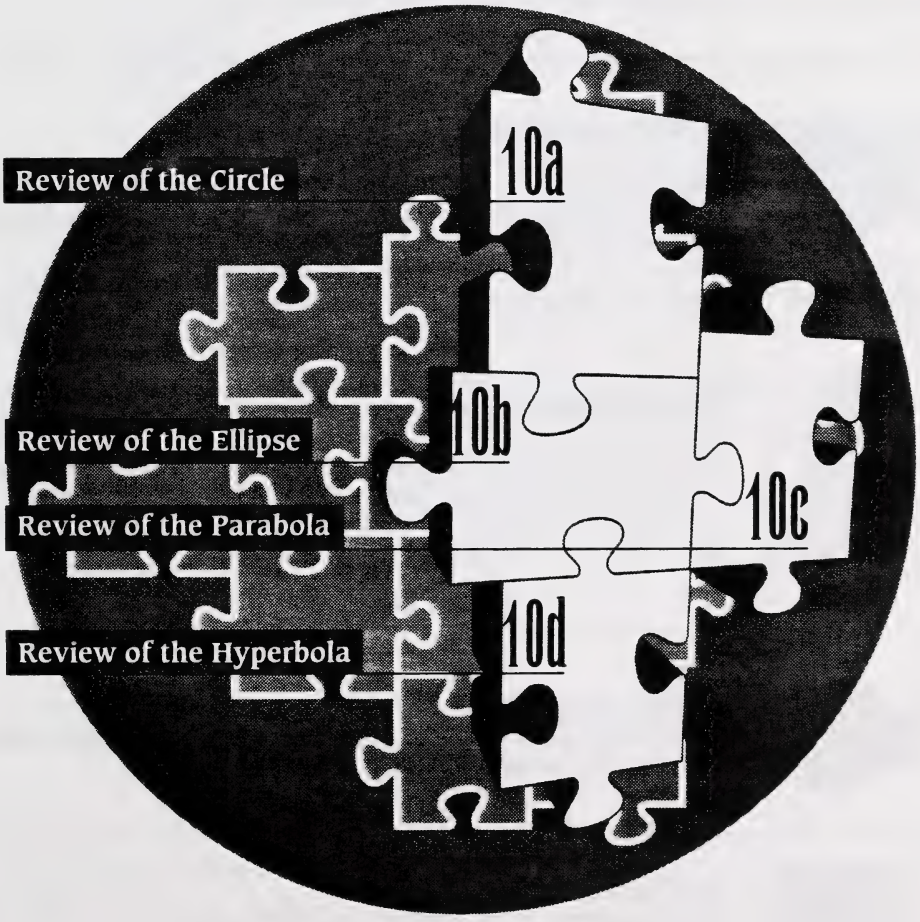
Lesson 10: REVIEW

LEARNER EXPECTATIONS

By the end of Lesson 10 you should be able to

- review and expand previously developed concepts with an emphasis on problem solving

This lesson has four parts:



EXPLANATION OF ICONS

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Review of the Circle

Watch the Introduction and Video Segment 10a.

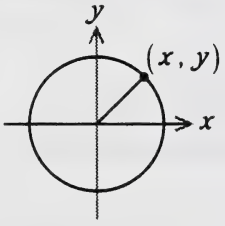
Recall the following:

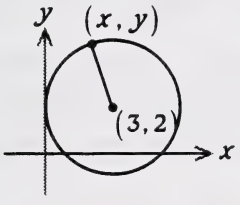
- The cutting plane intersects the cone perpendicular to the axis.
- The circle degenerates to a point or to no graph. (Only the case of one point can be drawn.)
- In the general equation, $Ax^2 + Cy^2 + Dx + Ey + F = 0$, A must equal C.
- The eccentricity is not defined.
- The circle is the locus of all points equidistant from a fixed point.

Changing the Coefficients *D*, *E* and *F* in the Equation of the Circle

A change in the absolute value of D and/or E will result in a horizontal and/or vertical shift and a change in the size of the circle – the larger the absolute value, the larger the circle.

If it is desired to move the circle right, left, up or down without changing its size, then as the absolute values of D and E increase, the value of F must increase.





For a circle of radius 3 centred at the origin and using the locus definition and the distance formula, the equation would be as follows:

$$(x-0)^2 + (y-0)^2 = 9$$

$$x^2 + y^2 - 9 = 0$$

Thus, $D = 0$, $E = 0$ and $F = -9$.

If the circle is now moved to a centre $(3, 2)$, but the size remains the same, the equation becomes

$$(x-3)^2+(y-2)^2=9$$

$$x^2+y^2-6x-4y+4=0$$

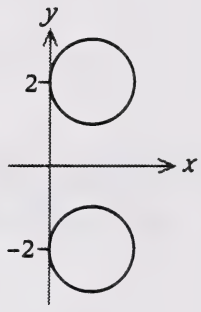
Thus, $D=-6$, $E=-4$ and $F=4$.

The value of F has increased from -9 to 4 as the absolute value of D increased from 0 to 6 and the absolute value of E increased from 0 to 4. Similarly, if the absolute values of D and E decreased, then the value of F would have to decrease in order for the circle to remain the same size.

In Video Segment 10a, the following question is asked: "Which coefficients change in the equations for the two circles shown?" The student answers that a reflection about the x-axis does not change the value of F.

The value of E changes sign but not its absolute value and, therefore, F does not change.

For the circle, as F decreases, the radius increases. As F increases, the radius decreases and the circle may ultimately first degenerate to a single point and then to no graph.



Example

A student graphs the circle defined by $x^2+y^2+2x-3y-49=0$. The student then changes the coefficient of x to 4 and the coefficient of y to -5, and graphs the equation again. Which of the following is true about the circle?

A. The centre moves up and left and the radius changes.
B. The centre moves down and right and the radius changes.
C. The centre moves up and left and the radius remains the same.
D. The centre moves down and right and the radius remains the same.

Solution

Increasing the D coefficient moves the centre of the circle to the left. Decreasing the E coefficient moves the centre up. Increasing the absolute value of either coefficient will increase the radius of the circle. Therefore, the correct solution is A.

Exercises

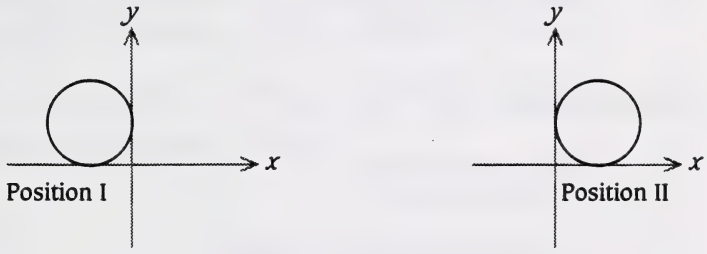
1. The graph of $3x^2+3y^2=0$ is

   A. a circle of radius 3 centred at the origin
   B. a circle of radius 3 centred at $(3, 3)$
   C. a single point at $(3, 3)$
   D. a single point at the origin

2. The equation $2x^2 + 2y^2 - 2x + 3y - 45 = 0$ defines a circle that has a centre in quadrant

   A. 1
   B. 2
   C. 3
   D. 4

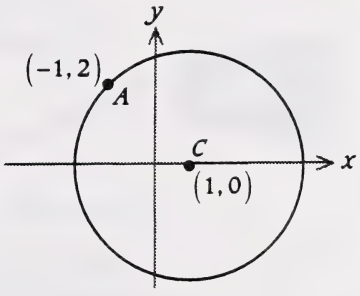
3. Joan wants to move a circle with equation $x^2 + y^2 + Dx + Ey + F = 0$ from Position I to Position II without changing its radius.



   What coefficient(s) must be changed?

   A. D and E
   B. E and F
   C. D only
   D. E only

4. a. If C is the centre of the circle, then determine the value of the radius.



   b. Determine the equation of the circle.

EXERCISE 10a SOLUTIONS are at the back of this workbook.

Review of the Ellipse

Challenge yourself

If the graph of $9x^2 + y^2 + Dx + Ey + F = 0$ is the graph of $9x^2 + y^2 - 25 = 0$ moved three units to the right, then explain what conditions the values of D, E and F must satisfy and why.

Recall the following:

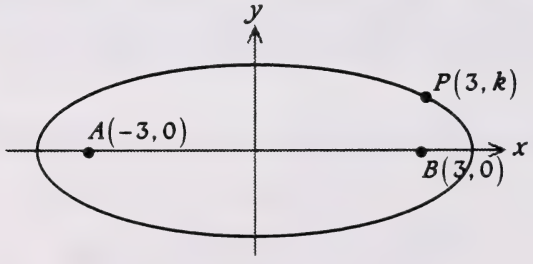
- The cutting plane intersects the cone at an angle greater than the angle of the generator but less than 90° when measured from the axis.
- The ellipse degenerates to a point or no graph. (Only the case of one point can be drawn.)
- In the general equation, $Ax^2 + Cy^2 + Dx + Ey + F = 0$; $A \neq C$; A and C must have the same sign.
- The eccentricity is between 0 and 1.
- The ellipse is the locus of all points, P, such that the sum of the distances from P to two fixed points is a constant.
- When $A = C$ in the general equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$, a circle is formed and is a special case of the ellipse.

MATH FACTOR

Watch Video Segment 10b.

Example

In the following diagram, the ellipse has focal points at A and B. If the ellipse has a constant sum of 10, then determine the value of k.



Solution

$$\begin{aligned}
\left|\overline{PA}\right| + \left|\overline{PB}\right| &= 10 \\
\sqrt{\left[3-(-3)\right]^2 + (k-0)^2} + \sqrt{(3-3)^2 + (k-0)^2} &= 10 \\
\sqrt{36+k^2} &= 10-k \\
36+k^2 &= 100-20k+k^2 \\
36 &= 100-20k \\
k &\doteq 3.2
\end{aligned}$$

Exercises

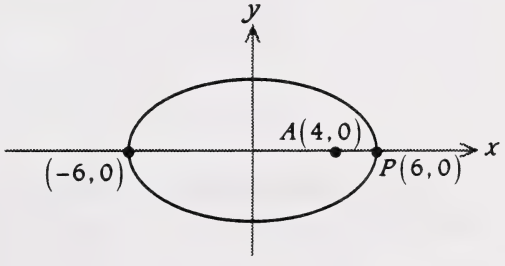
5. If the equation $3x^2 + Cy^2 + 36x - 45y - 56 = 0$ describes an ellipse, then which of the following is a possible value for C?

   a. -2
   b. 0
   c. 2
   d. 3

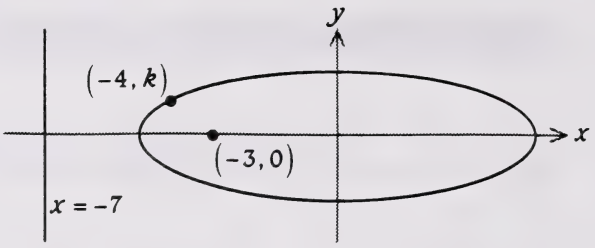
6. How does the graph of the equation $6x^2 + 3y^2 + 5y - 50 = 0$ change when the coefficient of y is 10?

   a. The graph will degenerate to a point.
   b. The graph will move upwards and increase in size.
   c. The graph will move downwards and increase in size.
   d. The graph will move upwards and remain the same size.
   e. The graph will move downwards and remain the same size.

7. If A is one of the foci in the ellipse at the right, then show that the constant sum is equal to the distance between the x-intercepts.



8. a. Calculate the value of k given that the ellipse in the diagram has an eccentricity of 0.5, a focus at $(-3, 0)$ and a directrix at $x = -7$.



b. Using the eccentricity definition, find the equation of the ellipse.

EXERCISE 10b SOLUTIONS are at the back of this workbook.

Review of the Parabola

The focus of a parabola is at the point $(10, 2)$ and the directrix is the line $x = 2$. Determine the equation of the locus.

Recall the following:

- The cutting plane intersects the cone parallel of the generator.
- The parabola can degenerate to two parallel lines, one line or no graph.
- In the general equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$, either $A = 0$ or $C = 0$.
- The eccentricity is equal to 1.
- The parabola is the locus of all points equidistant from a given line and a given point.

Example

In order for the parabola $Ax^2 + Cy^2 + Dx + Ey + F = 0$ to be nondegenerate with vertex at the origin and with a horizontal axis of symmetry, which coefficients must equal 0?

A. C, D and E
B. A, D, E and F
C. C, D and F
D. A, E and F

Solution

Since the axis of symmetry is horizontal, there must be a y^2-term. Therefore, $A = 0$. To be nondegenerate, there must be an x-term, thus, $D \neq 0$. If the vertex is on the origin, E and F must be 0. Therefore, D is the correct solution.

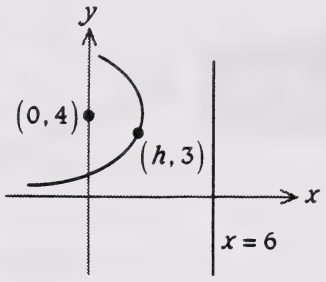
Exercises

9. Which of the following changes will not cause the parabola $2x^2 - 4x - 2y + 8 = 0$ to degenerate?

   A. Change the coefficient of x^2 to 0.
   B. Change the coefficient of x to 0.
   C. Change the coefficient of y and the constant term to 0.
   D. Change the coefficient of x^2 and the constant term to 0.

10. For each of the choices (A, B, C and D) in question 9, describe the parabola or degenerate that would be produced by each change.

11. In the graph of the parabola, the focus is at $(0, 4)$ and the directrix is $x = 6$.



a. Determine the value of h.

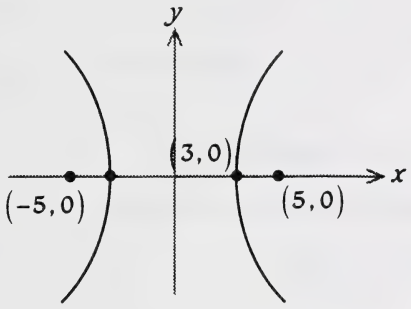
b. Determine the coordinates of the vertex.

c. Determine the equation of the parabola.

EXERCISE 10c SOLUTIONS are at the back of this workbook.

Review of the Hyperbola

Determine the constant difference for the hyperbola shown if its focal points are $(\pm 5, 0)$.



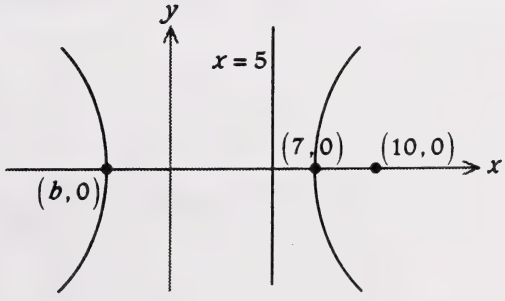
Recall the following:

- The cutting plane intersects the cone at an angle between the angle of the generator and the angle of the axis.
- The hyperbola degenerates to a pair of intersecting lines.
- In the general equation, $Ax^2 + Cy^2 + Dx + Ey + F = 0$, A and C must have different signs.
- The hyperbola is the locus of all points, P, such that the difference of the distances from P to two fixed points is constant.

Example

Determine the following for the graph of the hyperbola with focus at $(10, 0)$ and directrix at $x = 5$.

- the eccentricity
- the value of b
- another focus and directrix that could be used to generate the same graph



Solution

The eccentricity can be found using $(7, 0)$ as P, a point on the locus.

$$e = \frac{|\overline{PF}|}{|\overline{PD}|}$$
$$= \frac{3}{2}$$
$$= 1.5$$

To find the value of b, use $(b, 0)$ as a point on the locus.

$$\frac{|\overline{PF}|}{|\overline{PD}|} = \frac{3}{2} \qquad 15 - 3b = 20 - 2b$$
$$= \frac{10 - b}{5 - b} \qquad b = -5$$

Another directrix must be two units to the right of b: $x = -3$ and a second focus must be three units to the left of b $(-8, 0)$ because of the symmetry of the hyperbola.

Exercises

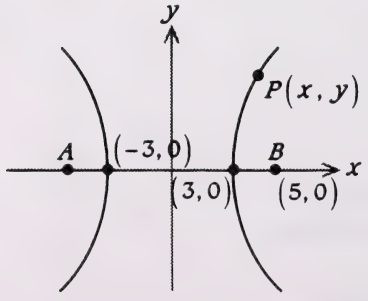
12. Which way will the centre of the hyperbola $x^2 - y^2 + 3x - 4y - 50 = 0$ move if the coefficient of y is changed to 4?

    A. It will move down.
    B. It will move up.
    C. It will move left.
    D. It will move right.

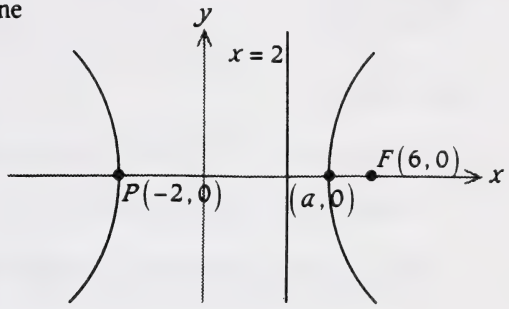
13. In order for the hyperbola $Ax^2 - Cy^2 + F = 0$ to degenerate to a pair of intersecting lines, what must be true?

    A. $A = 0$
    B. $C = 0$
    C. $F = 0$
    D. $F > 0$

14. In the following diagram, $|\overline{PB}| = 3$. Determine $|\overline{PA}|$.



15. For the hyperbola on the right, determine the following:



a. the eccentricity

b. the value of a

c. another focus and directrix that will generate the same graph

d. the equation

EXERCISE 10d SOLUTIONS are at the back of this workbook.

WHAT HAVE YOU LEARNED?
Return to the Learner Expectation at the beginning of this lesson and check it off if you have mastered it. A thorough knowledge of the conic sections is needed in such areas as projectiles, architecture, astronomy and space science.

Solutions

EXERCISE 10a

1. The answer is D. $(0,0)$ is the only point that will satisfy this equation. $3(0)^2 + 3(0)^2 = 0$

2. The answer is D since $D < 0$ and $E > 0$.

3. The answer is C. D must have its sign changed from a positive to a negative. Since there is no change in the absolute value of D, the size of the circle remains the same.

4. a. Find $|\overline{AC}|$.

$$|\overline{AC}| = \sqrt{(-1-1)^2 + (2-0)^2}$$
$$= \sqrt{2^2 + 2^2}$$
$$= \sqrt{8}$$
$$= 2\sqrt{2}$$

b. $(x-1)^2 + y^2 = 8;\ x^2 + y^2 - 2x - 7 = 0$

EXERCISE 10b

5. The answer is C. The coefficient of $C = 2$ is the same sign but a different value compared to $A = 3$.

6. The answer is C. The graph will move down as E has increased and will expand in size since the absolute value of E has increased.

7. The other focal point is at $B(-4,0)$. Then $|\overline{PB}| = 10$ and $|\overline{PA}| = 2$, resulting in a constant sum of 12. The distance between the x-intercepts is also 12 (6 + 6).

=8. a. $\dfrac{|\overline{PF}|}{|\overline{PD}|} = \dfrac{\sqrt{(-4+3)^2 + (k-0)^2}}{\sqrt{[-4-(-7)]^2 + (k-k)^2}}$

$$= \frac{1}{2}$$

$$2\sqrt{1+k^2} = \sqrt{9}$$
$$k^2 = 1.25 \text{ or } k \doteq 1.1$$

b. $\dfrac{|\overline{PF}|}{|\overline{PD}|} = 0.5 \qquad \dfrac{\sqrt{(x+3)^2 + y^2}}{\sqrt{(x+7)^2}} = \dfrac{1}{2}$

$$4\left(x^2 + 6x + 9 + y^2\right) = x^2 + 14x + 49$$
$$4x^2 + 24x + 36 + 4y^2 = x^2 + 14x + 49$$
$$3x^2 + 4y^2 + 10x - 13 = 0$$

EXERCISE 10c

9. The answer is B. In this case, the equation would be $2x^2 - 2y + 8 = 0$ or $y = x^2 + 4$, which is a nondegenerate parabola.

10. A – straight line: $y = -2x + 4$

    B – parabola: $y = x^2 + 4$

    C – two parallel vertical lines: $2x^2 - 4x = 0$; $x = 0, x = 2$

    D – straight line: $-4x - 2y = 0$ or $y = -2x$

11. a. $|\overline{PF}| = |\overline{PD}|$

$$\therefore \sqrt{(h-0)^2 + (3-4)^2} = \sqrt{(6-h)^2 + (3-3)^2}$$
$$\sqrt{h^2 + 1} = 6 - h$$
$$h^2 + 1 = 36 - 12h + h^2$$
$$12h = 35$$
$$h \doteq 2.92$$

    b. The vertex is $(3, 4)$ (halfway between $(0, 4)$ and the line $x = 6$).

    c. The equation is $y^2 + 12x - 8y - 20 = 0$.

EXERCISE 10d

12. The answer is E. Since E has increased and $C < 0$, the conic will move up instead of down.

13. The answer is C. This will produce $Ax^2 - Cy^2 = 0$ or $y = \pm\sqrt{\frac{A}{C}}x$, a pair of intersecting lines.

14. The constant difference will be $|\overline{AQ}| - |\overline{QB}| = 8 - 2 = 6$.

$$\therefore |\overline{PA}| - |\overline{PB}| = 6$$
$$|\overline{PA}| - 3 = 6$$
$$|\overline{PA}| = 9$$

15. a. $|\overline{PF}| = 6-(-2)$ $\quad |\overline{PD}| = 2-(-2)$

$= 8$ $\quad = 4$

$\therefore e = 2$

b. $2 = \dfrac{6-a}{a-2}$

$\therefore 2a-4 = 6-a$ and $a = 3\frac{1}{3}$

c. The focus must be $2\frac{2}{3}$ to the left of P: $\left(-4\frac{2}{3}, 0\right)$

The directrix must be $1\frac{1}{3}$ units to the right of P: $x = -\frac{2}{3}$

d. The equation is $3x^2 - y^2 - 4x - 20 = 0$.